FLUID DYNAMICS

CIRCULATION AND VORTICITY (DEFINITION AND PROBLEMS)

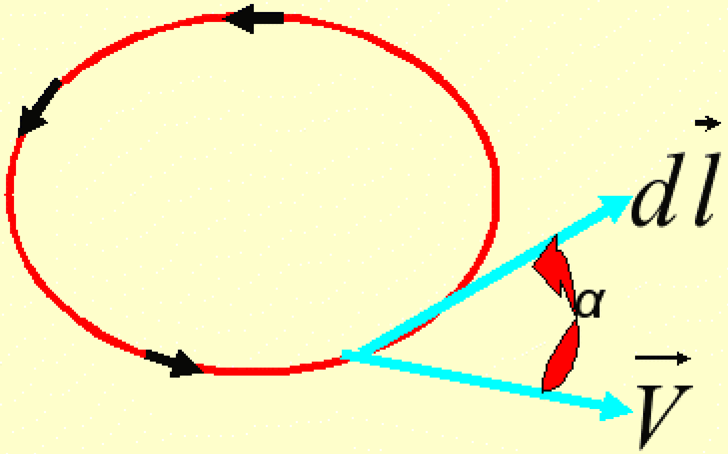
Circulation

Circulation = **macroscopic measure of rotation for a finite area of the fluid**

$$C = \oint \vec{V} \bullet d\vec{l} = \oint \left|\vec{V}\right| \cdot \cos\alpha dl$$

$d\vec{l}$ - element of the contour

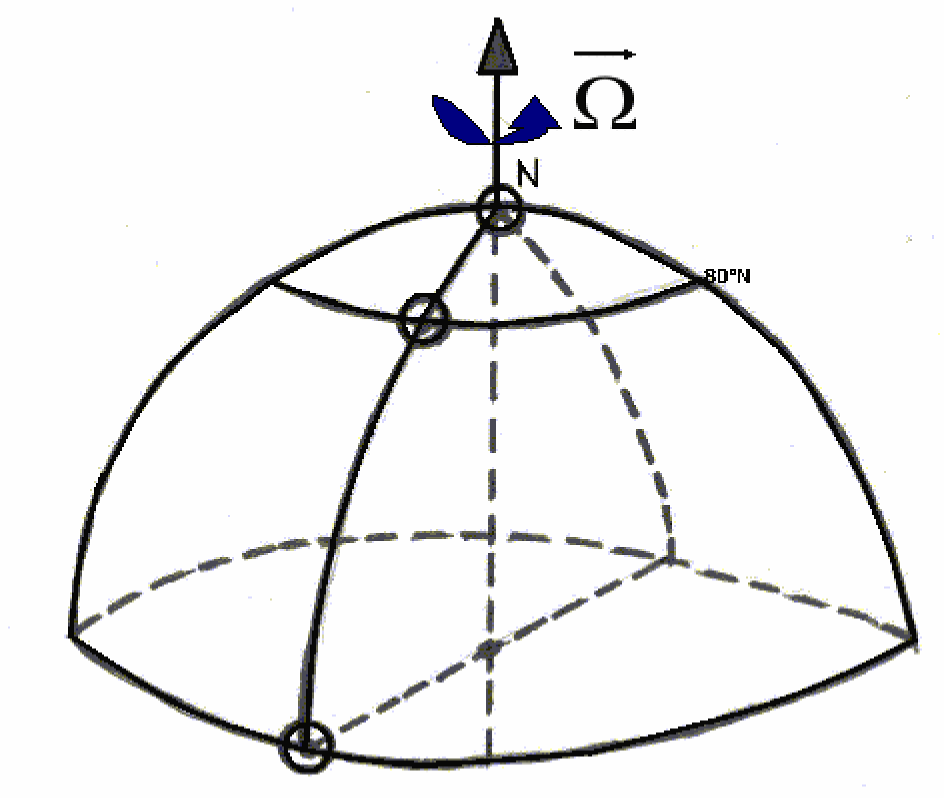
(vector tangent to the contour)



Convention: C > 0 for counter-clockwise integration around the contour

Problem:

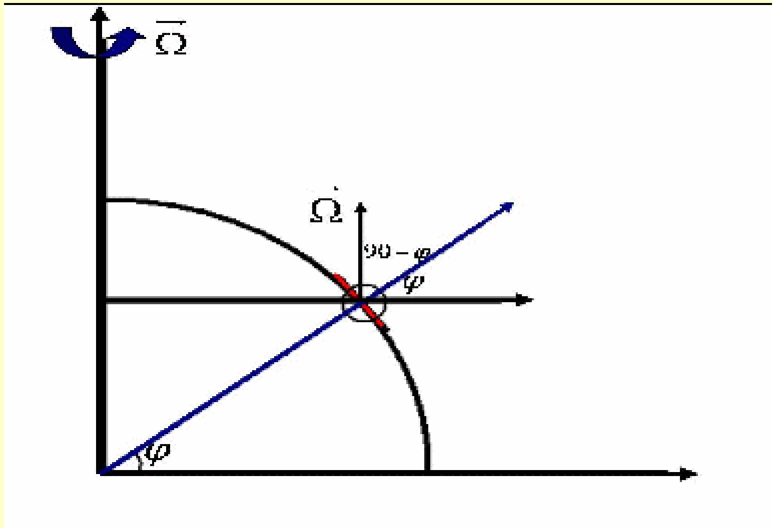
Suppose that the Earth's atmosphere rotates solidary (motionless relative to the Earth's surface) with the Earth around its rotation axis. The period of rotation is 24h. Calculate the circulation of air along a circular contour of radius R=100km, centred at 90°N, 80°N, 70°N, 60°N, 50°N, 40°N, 30°N, 20°N, 10°N and 0°.



Write the results in the following table as well as a plot of C(φ)

φ	C (m^2/s)
90	
80	
70	
60	
50	
40	
30	
20	
10	
0	

SOLUTION



$$\Omega = 2\cdot\pi / T = (2\cdot 3.14)/(24\cdot 3600) = 7.26\cdot 10^{-5}\, s^{-1}$$

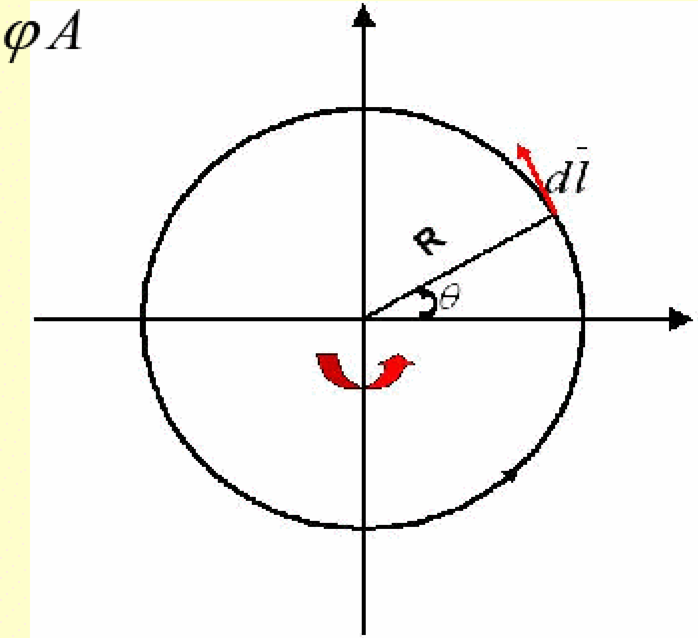
$$C = \oint \left|\overrightarrow{V}\right| \cos\alpha \, dl$$

$$\left|\overline{V}\right| = V = \Omega R \sin\varphi$$

$$\alpha = 0 \rightarrow \quad \cos\alpha = 1$$

$$dl = R\, d\Theta$$

$$C = \int_0^{2\pi} \Omega R \sin\varphi \cdot 1 \cdot R \cdot d\theta = 2\pi R^2 \sin\varphi\, \Omega = 2\Omega \sin\varphi\, A$$



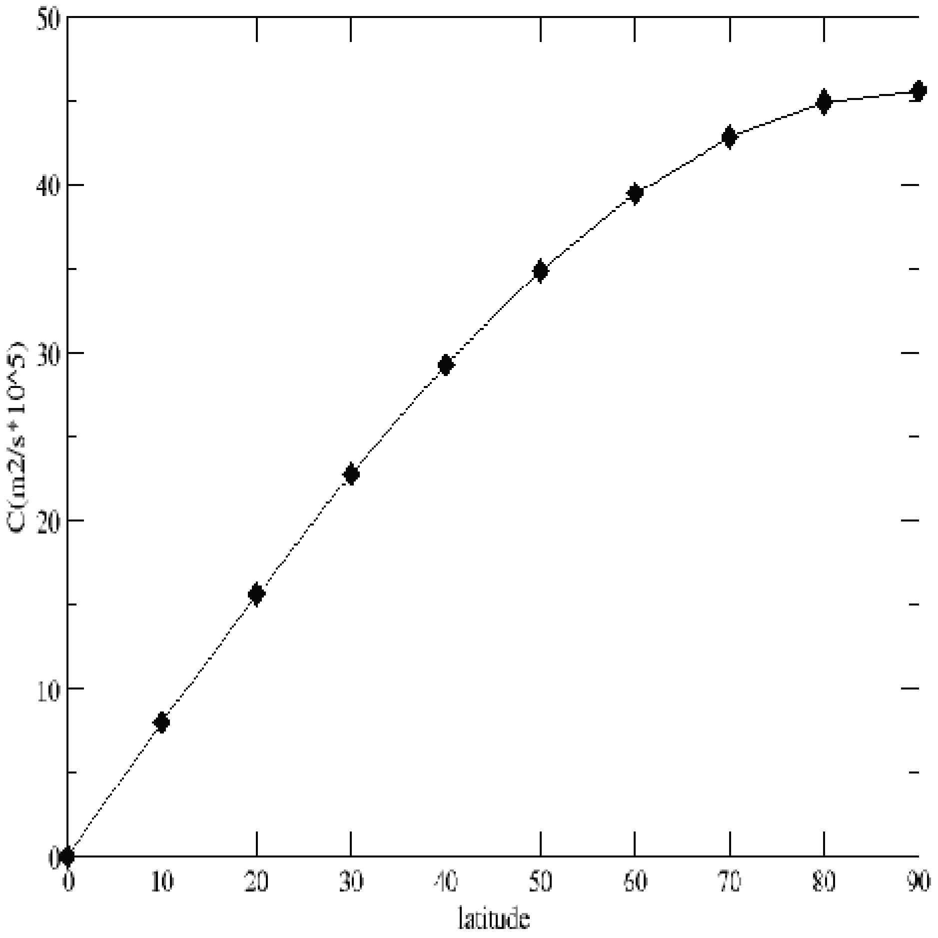
$$A = \pi R^2 \rightarrow \quad C = 2\Omega \sin\varphi\, A$$

$$C = k\cdot\sin\varphi$$

$$k = 2\Omega A$$

$$k = 2\cdot 7.26\cdot 10^{-5}\cdot 3.14\cdot 10^{10} = 45.59\, m^2/s$$

φ	$C\,(10^5\,m^2/s)$
90	45.59
80	44.89
70	42.84
60	39.48
50	34.92
40	29.30
30	22.79
20	15.59
10	7.91
0	0



Vorticity

Vorticity – a microscopic measure of the rotation at any point in the fluid (field).

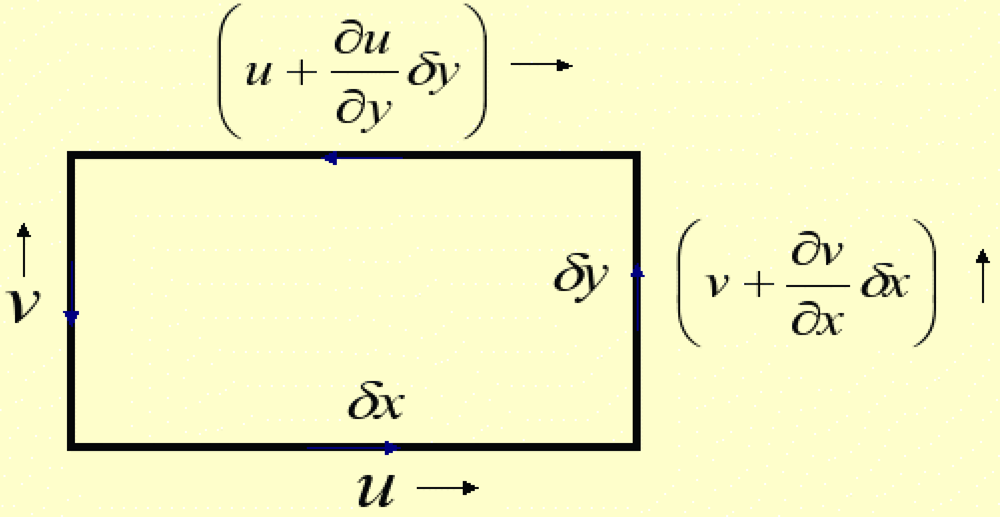
$$\vec{\omega} = \nabla \times \vec{V} = rot\left(\vec{V}\right)$$

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \rightarrow velocity$$

For **horizontal movements**, the vorticity has only the **vertical component:**

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- **Vorticity and circulation are related**
- **Consider a closed rectangular contour oriented counter-clockwise**



Circulation along the contour is:

$$\delta C = u\,\delta x + \left(v+\frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u+\frac{\partial u}{\partial y}\delta y\right)\delta x - v\,\delta y$$

$$\delta C = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\delta x\,\delta y = \xi\,\delta A \quad \rightarrow \quad \boxed{\xi = \lim_{\delta A\to 0}\frac{\delta C}{\delta A}}$$

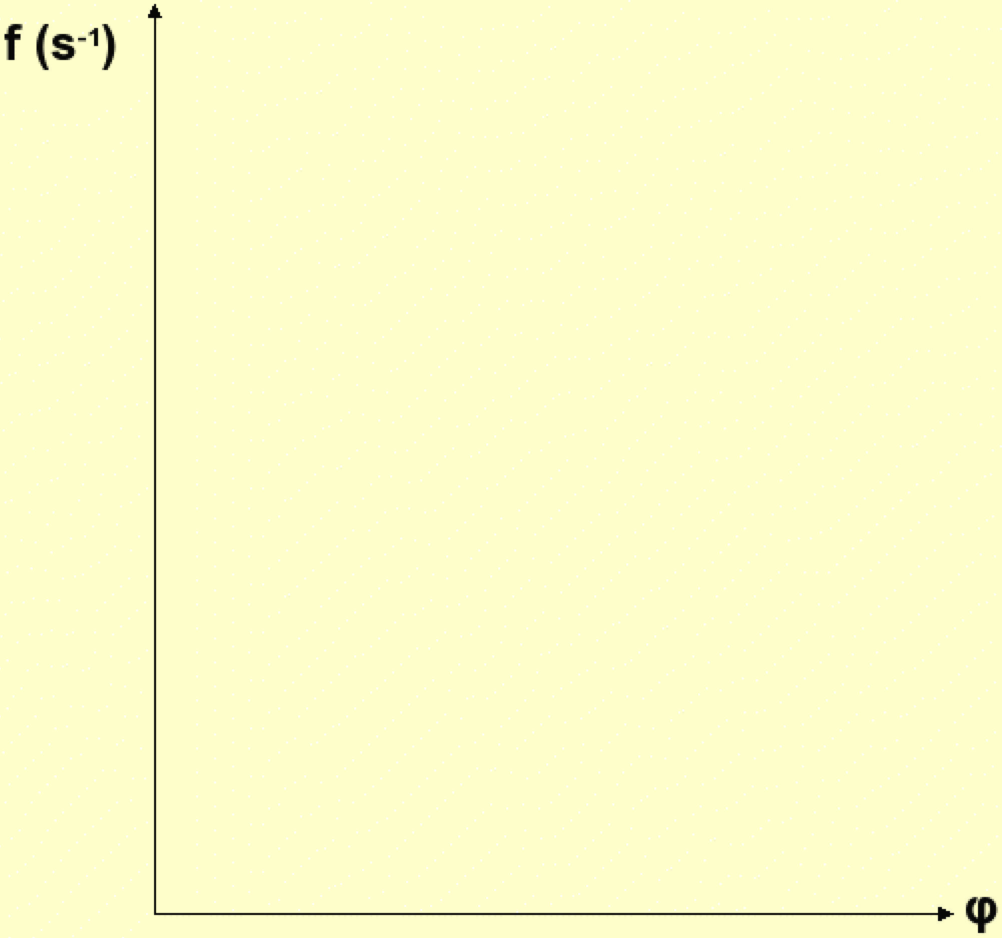
$$\boxed{\oint \vec{V}\cdot d\vec{l} = \iint_A \left(\nabla\times\vec{V}\right)\cdot\vec{n}\,dA}$$

$\vec{n}$ -unit vector normal to the area element dA

PROBLEM

Calculate the vorticity of the air inside the circulation contour as defined in the previous problem. Arrange the result as a table, as well as a graphic.

φ	$f(s^{-1})$
90	
80	
70	
60	
50	
40	
30	
20	
10	
0	



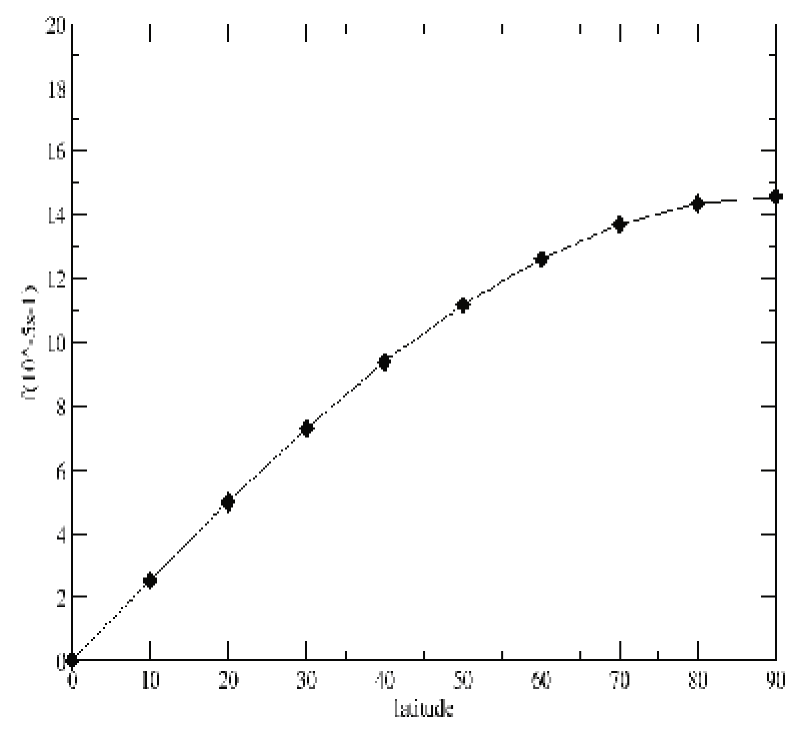
Solution

In this case vorticity is the circulation along the contour divided by the area area enclosed by the contour.

$$f = \frac{C}{A} = \frac{2\Omega \sin\varphi A}{A} = 2\Omega \sin\varphi$$

f – Coriolis parameter (planetary vorticity)

φ	$f\,(10^{-5}\,s^{-1})$
90	14.58
80	14.35
70	13.70
60	12.62
50	11.16
40	9.37
30	7.29
20	4.98
10	2.53
0	0



Shear vorticity and curvature vorticity

Physical interpretation of vorticity may be facilitated using the natural coordinate system.

We can consider an infinitesimal contour in the fluid:

$$\delta C = V\left[\delta S + d\left(\delta S\right)\right] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta S$$

Rs-curvature radius

$\vec{n}$ -unit vector indicating curvature center

$$but: d\left(\delta S\right) = \delta\beta\delta n$$

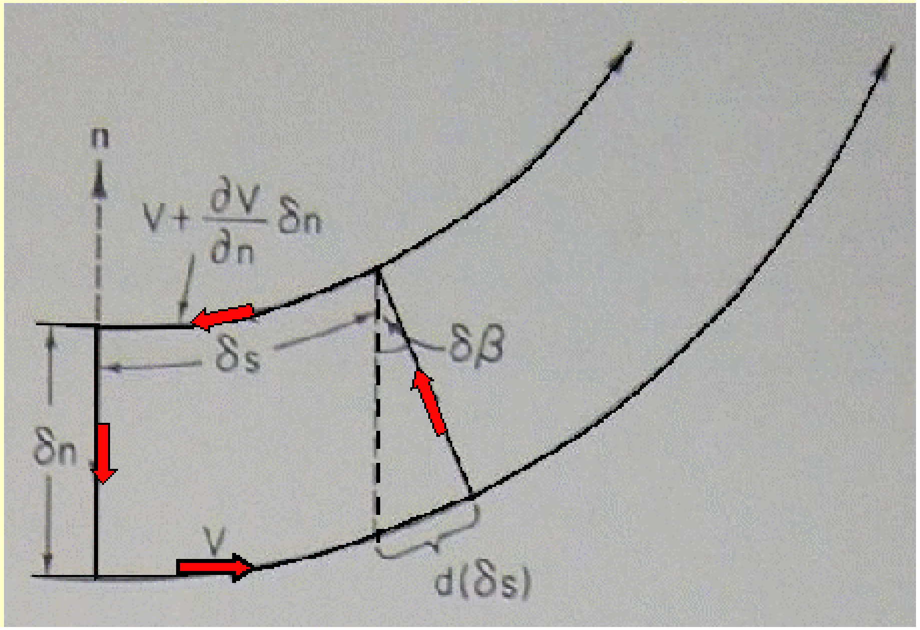
δβ **– angular change in the fluid direction in the distance δ*S***

$$\rightarrow \delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta S}\right)\delta n\,\delta S$$

$$\boxed{\xi = \lim_{\delta n,\delta S \rightarrow 0}\frac{\delta C}{\left(\delta n\,\delta S\right)} = -\frac{\partial V}{\partial n} + \frac{V}{R_S}}$$

Where: $-\frac{\partial V}{\partial n}$ – *shear vorticity*

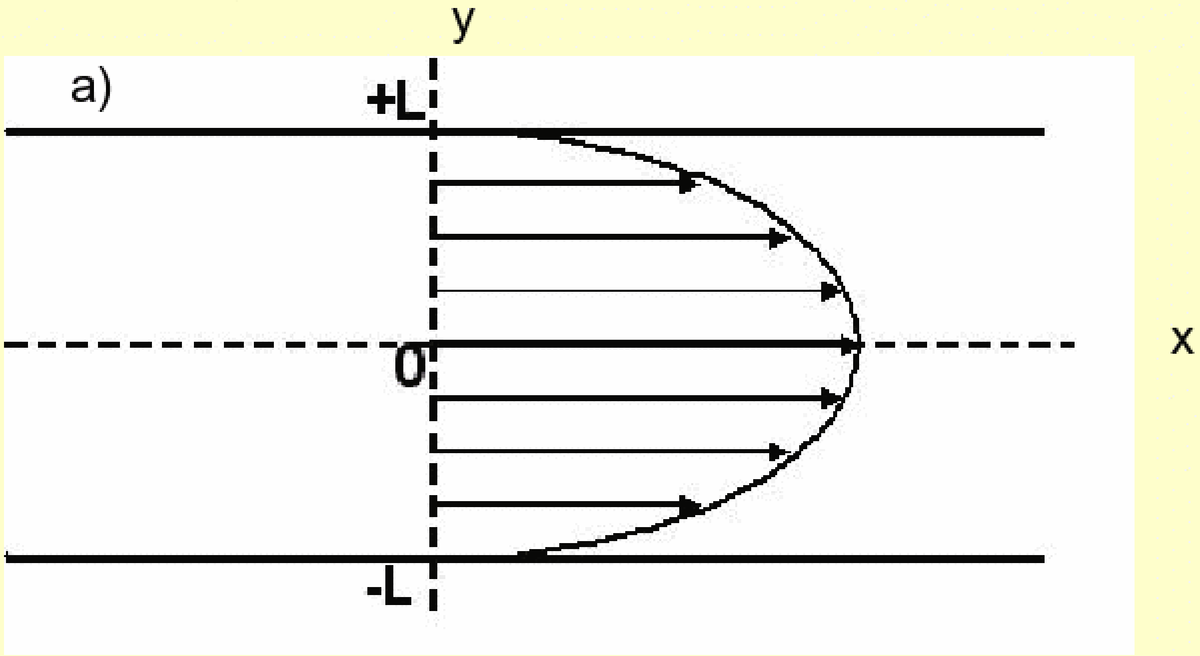
$\frac{V}{R_S}$ – *curvature vorticity*

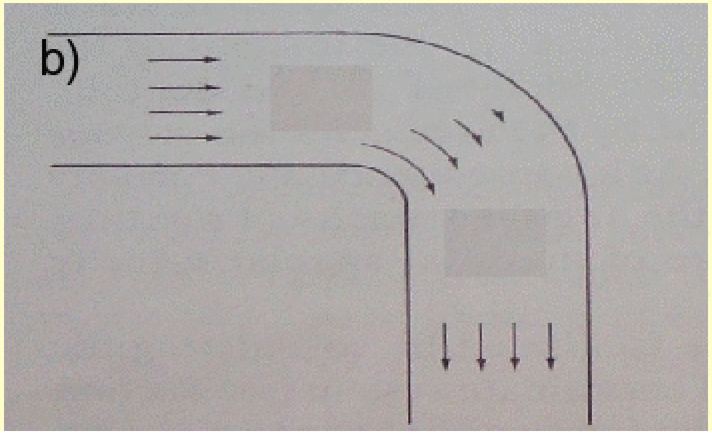


Problem

A viscous fluid flows in a pipe of diameter D=2L(figure a). The velocity at the boundary is zero due to friction, and increases towards the centre of the pipe. What is the sign of the shear vorticity in the upper and lower part of the pipe? Calculate the vorticity if the velocity profile is:

$$u = U\,\mathbf{cos}\left(\frac{y}{L}\cdot\frac{\pi}{2}\right)$$





What is the shear and curvature vorticity of the fluid (figure b) in different sections of the pipe?

SOLUTION

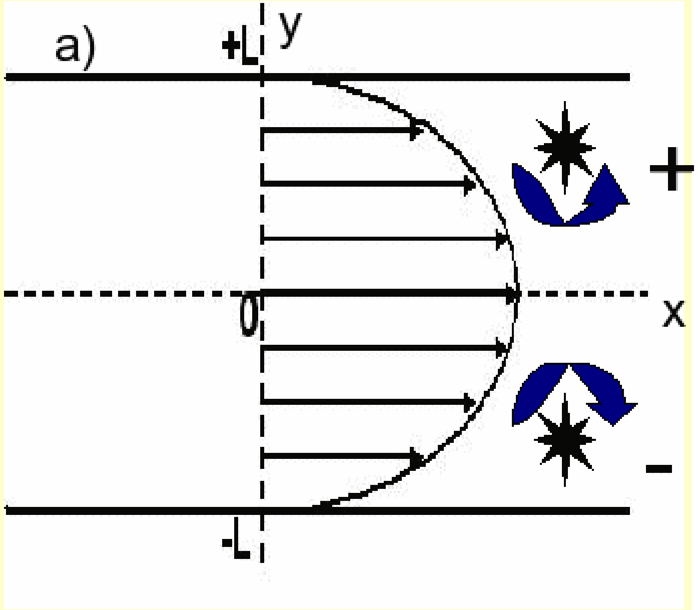
$$R_S = 0 \rightarrow curvature\,vorticity = 0$$

$$\frac{\partial u}{\partial y} \neq 0 \rightarrow shear\ vorticity \neq 0 \begin{cases} +in\,the\,upper\,part \\ -in\,the\,lower\,part \end{cases}$$

$$\xi = -\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\left(U \cdot \cos\left(\frac{y}{L} \cdot \frac{\pi}{2}\right)\right) = \frac{\pi}{2L} U \sin\left(\frac{y}{L} \cdot \frac{\pi}{2}\right)$$

$$y \in (0, L) \Rightarrow \xi > 0$$

$$y \in (-L, 0) \Rightarrow \xi < 0$$



Curved flow may have zero vorticity due to the fact that the shear vorticity is equal and opposite to the curvature vorticity. In our case the fluid along the inner margin of the pipe flows faster in just the right proportion so that the paddle wheel does not run.