Barotropy and baroclinicy

The vertical distribution of mass in the atmosphere can be represented by a collection of constant specific volume surfaces separated by layers of one specific volume thickness. These surfaces $\alpha=1/\rho=$constant are referred to as ISOSTERIC surfaces. Analogously, the vertical pressure distribution can be represented by a collection of constant pressure (isobaric) surfaces separated by layers of one pressure unit thickness. Since the slope of isobaric surfaces is very small, they can be considered to be almost horizontal.

The isosteric and isobaric surfaces usually intersect in space defining a continuous family of unit isosteric-isobaric surfaces, which are called isosteric-isobaric solenoids. In figure 1a you can see a cross-section in a baroclinic atmosphere. The number of tubes per unit area expresses the intensity of the solenoidal field.

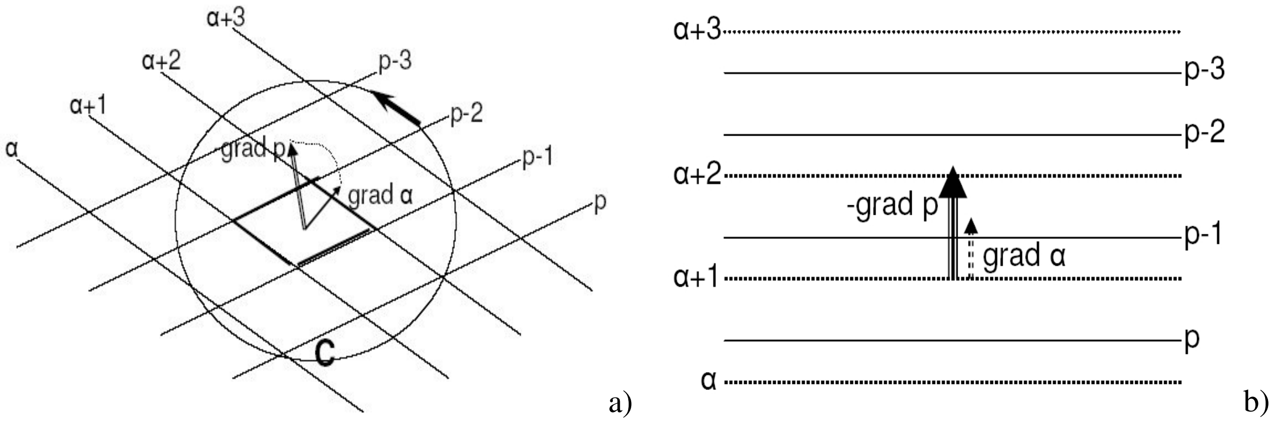

Figure 1. *Isosteric-isobaric solenoids*

a) Baroclinic atmosphere

b) Barotropic atmosphere

In mathematical terms this is given by the magnitude of the baroclinicity vector N, defined as:

$$N = (\text{grad } \alpha) \text{ x } (-\text{grad } p) = - \text{curl } (\alpha \text{ grad } p) \quad (1)\text{, taking into account that grad } p = 0.$$

In meteorology and especially in the synoptic applications, the isosteric-isobaric applications are seen mostly in terms of temperature and pressure:

$$N = (-\text{grad } T) \text{ x } (R/p \text{ grad } p) \qquad (2)$$

or in terms of potential temperature and temperature:

$$N = - c_p \text{ grad}(ln\Theta) \text{ x grad } T = -\text{grad } s \text{ x grad } T \qquad (3)$$

where s is the entropy.

In this case the stratification of the atmosphere is called *baroclinic*. The number of solenoids per unit area measures the degree of baroclinicity.

When N = 0 (no solenoids), the surfaces are parallel and the stratification is called *barotropic.*

In a simple way we can say that the atmosphere is barotropic when surfaces of constant specific volume (density, temperature or potential temperature) coincide with surfaces of constant pressure. The vectors grad α and grad p are parallel and proportional to each other so that the speicifc volume is just a function of pressure (e.g. the adiabatic atmosphere).

In general the atmosphere is baroclinic since the density depends on temperature as well as on pressure and specific humidity.

1. Barotropic instability

Baroclinic instability usually appears in the tropics region due to the lack of atmospheric fronts. Due to the fact that at the tropics there is a lack of condensation, vertical motions are small. Thus, to a first approximation the flow is governed by the barotropic vorticity equation:

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\bullet\nabla\right)(\xi+f)=0 \qquad (4)$$

We assume that the flow consists of a small barotropic perturbation superposed on a zonal current which depends only on latitude.

In this case we have:

$$u=\bar{u}(y)+u' \quad (5)$$

$$v = v' \quad (6)$$

Because the flow is quasi-nondivergent:

$$u'=-\frac{\partial\psi'}{\partial y}, \qquad v'=\frac{\partial\psi'}{\partial x} \qquad (7)$$

If we put (7) in (4) we get:

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi'+\left(\beta-\frac{d^2\bar{u}}{\partial y}\right)\frac{\partial\psi'}{\partial x}=0 \qquad (8)$$

It results that the term $\beta-\frac{d^2\bar{u}}{dy^2}=\frac{d}{dy}(f+\xi)$ (9) is the latitudinal gradient of the basic state absolute vorticity.

We assume further that the solutions of eq. 9 can be represented in terms of zonally propagating harmonic waves in the form:

$$\psi'(x,y,t)=\psi(y)e^{ik(x-ct)} \qquad (10)$$

Where $\Psi = \Psi_r + i\Psi_i$, is a complex function depending only on y alone.

Putting (10) in (8) we get:

$$(\bar{u}-c)\left(\frac{d^2\psi}{dy^2}-k^2\psi\right)+\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\psi=0 \qquad (11)$$

It is not a simple matter to determine the solutions of (11) for a particular profile u(y), because the coefficients are not constant.

It is possible to obtain "necessary" conditions for the existence of instability by application of simple integral considerations. If we divide (11) by $(\bar{u}-c)$ we obtain:

$$\frac{d^2\psi}{dy^2}-\left(k^2-\frac{\beta-\frac{d^2\bar{u}}{dy^2}}{\bar{u}-c}\right)\psi=0 \quad (12)$$

If the phase speed is complex: $c=c_r+ic_i$, then $(\bar{u}-c)^{-1}$ is also complex and has real and imaginary parts:

$$\delta_r=\frac{u-c_r}{(u-c_r)^2+c_i^2}, \qquad \delta_i=\frac{c_i}{(u-c_r)^2+c_i^2} \qquad (13)$$

Eq. (12) can be spitted into real and imaginary parts:

$$\frac{d^2\psi_r}{dy^2}-\left[k^2-\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_r\right]\psi_r-\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i\psi_i=0 \quad (14)$$

$$\frac{d^2\psi_i}{dy^2}-\left[k^2-\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i\right]\psi_i+\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i\psi_r=0 \quad (15)$$

Multiplying (14) by ψ_i, (15) by ψ_r and subtracting the latter from the former we obtain:

$$\psi_i\frac{d^2\psi_r}{dy^2}-\psi_r\frac{d^2\psi_i}{dy^2}-\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i\left(\psi_i^2+\psi_r^2\right)=0 \quad (16)$$

which can be written as:

$$\frac{d}{dy}\left(\psi_i\frac{d\psi_r}{dy}-\psi_r\frac{d\psi_i}{dy}\right)=\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i\left(\psi_r^2+\psi_i^2\right) \quad (17)$$

If we integrate (17) with respect to y and we apply the boundary conditions:

$\psi_i=\psi_r=0$ at $y=\pm L$ (18)

we find that the terms on the left integrate to zero, so that we are left with the integral condition

$$\int_{-L}^{+L}\left(\beta-\frac{d^2\bar{u}}{dy^2}\right)\delta_i|\psi|^2\,dy=0 \qquad (19)$$

where: $|\psi|^2=\psi_r^2+\psi_i^2$

An unstable perturbation to exist requires: $\delta_i > 0$ (i.e. $c_i > 0$).

Since $|\psi|^2 \geq 0$ everywhere in the domain, the integral condition in (19) can be satisfied for an unstable wave only if $\beta - \frac{d^2\bar{u}}{dy^2}$ changes sign somewhere in the region –L < y < L. Thus, a necessary condition for barotropic instability is that the gradient of absolute vorticity of the mean current must vanish somewhere in the region, that is:

$$\beta - \frac{d^2\bar{u}}{dy^2} = 0 \text{ somewhere.} \quad (20)$$

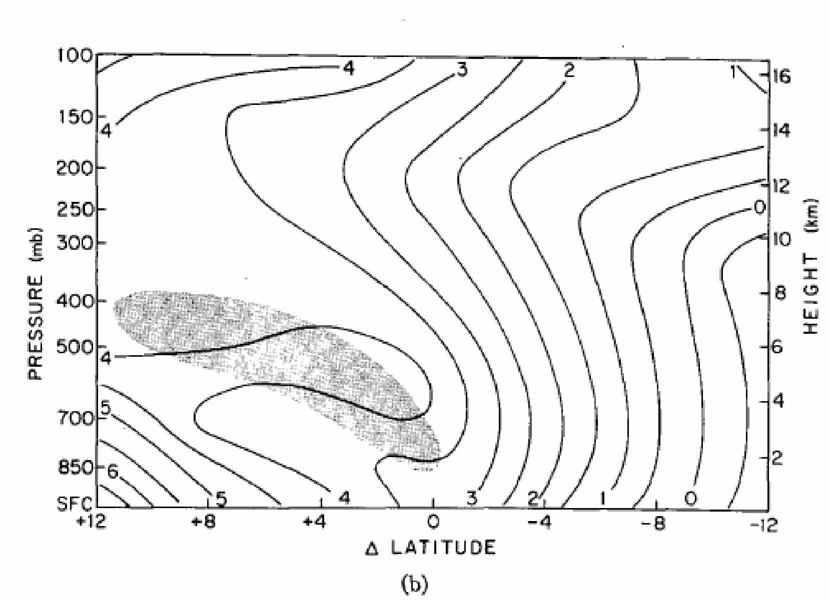

Figure 2. Absolute vorticity corresponding to the mean wind field for the African Jet. Shaded area shows region where $\beta - \bar{u}_{yy}$ is negative

The shaded region indicates the area in which the vorticity gradient is negative. It means that the African jet satisfies the necessary condition (20) for barotropic instability. Numerical solutions for the perturbation equation (12) for the 700-mb profile $\bar{u}(y)$ observed in the African jet indicate that the jet is in fact barotropically unstable. Barotropic instability appears to be the mechanism responsible for the generation of African wave disturbances.

Barotropic instability is also possible in the vicinity of the midlatitude jet stream. However, at middle latitudes baroclinic instability is generally the more important mechanism.

2. Baroclinic instability

In figure 2 you have the normal distribution of atmospheric temperature with respect to altitude (on the left) and latitude (on the right). Temperature typically decreases with altitude in the troposphere. Temperature also generally decreases with latitude from the warm equator to the cold poles.

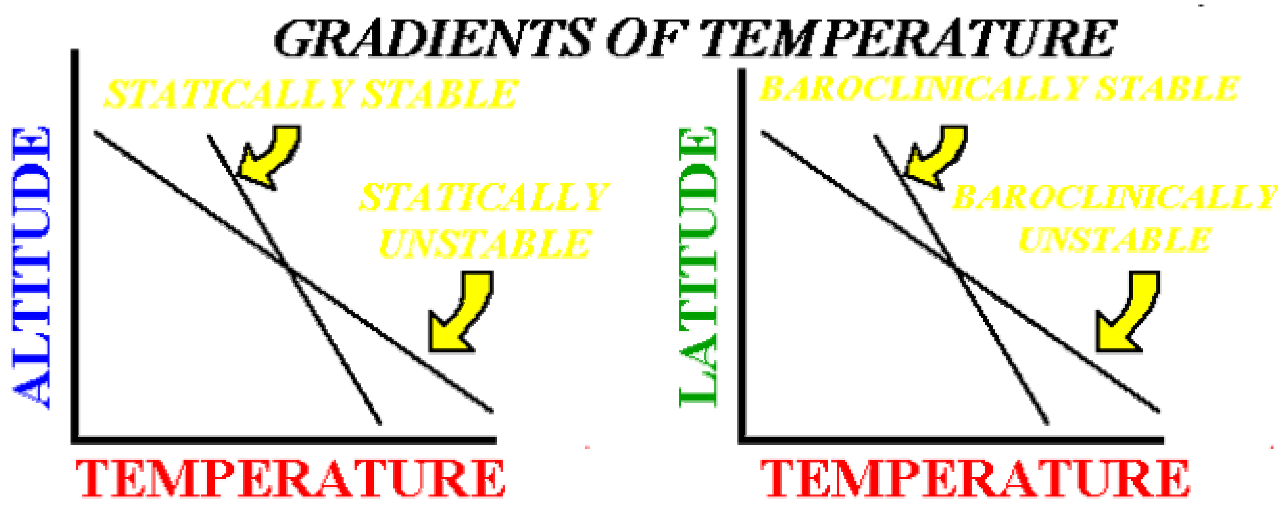

Figure 2. Normal distribution of atmospheric temperature with respect to altitude (on the left) and latitude (on the right)

Decreasing temperature with respect to altitude can lead to the development of convection, if the vertical temperature distribution becomes *statically unstable*. Static instability occurs when the vertical temperature gradient becomes too large, meaning that the air at low levels is too warm and the air at higher altitudes is too cold. When the atmosphere is in a statically unstable state, it will break down into convection so as to transport the excess energy away from the surface and upward into the cooler altitudes aloft. In an analogous manner, if the latitudinal distribution of temperature develops an equator-to-pole temperature gradient which is too large, that indicates that the tropics are too warm and the poles too cold. This equator-to-pole imbalance in energy is fundamentally due to the excess net radiational heating in tropical latitudes. Such an energy distribution is unstable, and the name given to this unstable state is *baroclinic instability*. In a state of baroclinic instability, the atmosphere takes on wind flow characteristics to move the excess energy from the regions of excess to regions of deficit.

In figure 3 the middle latitude westerly wind flow is shown in two different states. On the left is the typical wind regime under conditions of baroclinic *stability*, when the imbalance of energy between tropics and polar regions is not excessively large. While there is always an equator-to-pole temperature gradient which (based on laws of wind dynamics on a rotating planet) leads to mid-latitude westerlies, the westerly wind currents have no great need to transport excess energy in this state of relative stability.

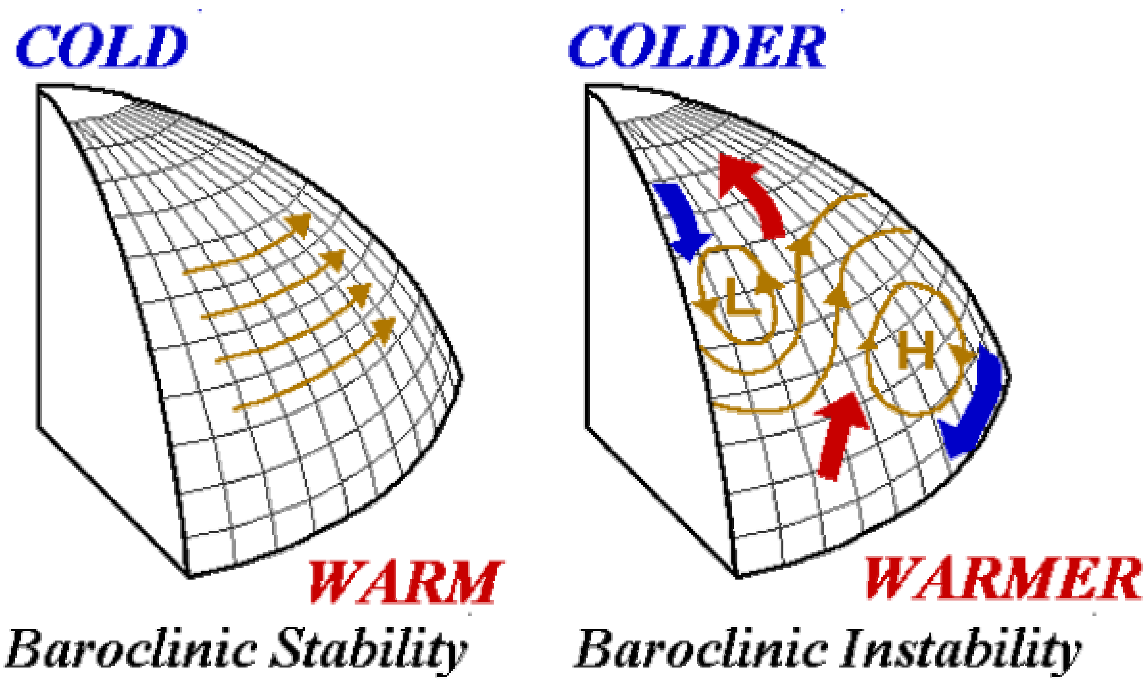

Figure 3. Middle latitude westerly wind flow

Since westerly wind flow parallel to latitude circles will not transport much energy poleward, the atmosphere in a baroclinically stable state will continue to build up the equator-to-pole temperature gradient under the influence of excess net radiational heating in the tropics. When the temperature gradient reaches an excessively large value, the atmosphere becomes baroclinically *unstable* and the wind currents respond by developing poleward energy transporting modes of flow. As seen in the sketch on the right, this is accomplished by the development of large meanders in the westerly flow, and cut-off pressure centers, which provide pathways for warm and cold air pools to move across latitudes, thus achieving the required energy transports.

The most important application of the baroclinic instability is the cyclogenesis process at the mid-latitudes. Cyclogenesis represents the development of synoptic scale weather disturbances. In baroclinic instability the potential energy of the basic state flow is converted to potential energy and kinetic energy of the perturbation. Further we will derive the conditions for baroclinic instability using the two-level quasi-geostrophic model, a model in which the geopotential is predicted at two levels.

The basic equations of the two-level model are:

$$\frac{\partial}{\partial t}\nabla^2 \psi_1 + \mathbf{V_1}\bullet\nabla\left(\nabla^2 \psi_1 + f\right) = \frac{f_0}{\Delta p}\omega_2 \qquad (21)$$

$$\frac{\partial}{\partial t}\nabla^2 \psi_3 + \mathbf{V_3}\bullet\nabla\left(\nabla^2 \psi_3 + f\right) = -\frac{f_0}{\Delta p}\omega_2 \qquad (22)$$

$$\frac{\partial}{\partial t}(\psi_1 - \psi_3) + \mathbf{V_2}\bullet\nabla(\psi_1 - \psi_3) = \frac{\sigma\Delta p}{f_0}\omega_2 \qquad (23)$$

where $\mathbf{V_j} = \mathbf{k}\times\nabla\psi_j$ for j=1,2,3

We assume that the stream functions ψ_1 and ψ_3 consist of basic state parts which depend only on y, plus perturbations which depends only on x and t:

$$\psi_1 = -U_1 y + \psi_1'(x,t)$$

$$\psi_3 = -U_3 y + \psi_3'(x,t) \qquad (24)$$

$$\omega_2 = \omega_2'(x,t)$$

$U_1 - \textit{zonal velocity at level } 1$

$U_3 - \textit{zonal velocity at level } 3$

In this case the perturbation field has meridional and vertical velocity components only.

Substituting (24) in (21)-(23) and linearizing we obtain the perturbation eq.:

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_1'}{\partial x^2} + \beta\frac{\partial\psi_1'}{\partial x} = \frac{f_0}{\Delta p}\omega_2' \qquad (25)$$

$$\left(\frac{\partial}{\partial t} + U_3\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_3'}{\partial x^2} + \beta\frac{\partial\psi_3'}{\partial x} = -\frac{f_0}{\Delta p}\omega_2' \qquad (26)$$

$$\left(\frac{\partial}{\partial t} + \frac{U_1 + U_3}{2}\frac{\partial}{\partial x}\right)(\psi_1' - \psi_3') - \frac{U_1 - U_3}{2}\frac{\partial}{\partial x}(\psi_1' + \psi_3') = \frac{\sigma\Delta p}{f_0}\omega_2' \qquad (27)$$

where β=df/dy (β-plane approximation)

We assume wave-type solutions:

$$\psi_1' = Ae^{ik(x-ct)}, \qquad \psi_3' = Be^{ik(x-ct)}, \qquad \omega_2' = Ce^{ik(x-ct)} \qquad (28)$$

Substituting (28) in (25)-(27) we find that the amplitude A, B and C must satisfy the following set of homogeneous and linear equations:

$$ik\left[(c - U_1)k^2 + \beta\right]A - \frac{f_0}{\Delta p}C = 0 \qquad (29)$$

$$ik\left[(c-U_3)k^2+\beta\right]B+\frac{f_0}{\Delta p}C=0 \qquad (30)$$

$$-ik(c-U_3)A+ik(c-U_1)B-\frac{\sigma\Delta p}{f_0}C=0 \qquad (31)$$

Since the set is homogeneous, nontrivial solutions will exist only if the determinant of the coeff. A, B and C is zero. This implies that the phase speed *c* satisfies:

$$\begin{vmatrix} ik\left[(c-U_1)k^2+\beta\right] & 0 & -f_0/\Delta p \\ 0 & ik\left[(c-U_3)k^2+\beta\right] & f_0/\Delta p \\ -ik(c-U_3) & ik(c-U_1) & -\sigma\Delta p/f_0 \end{vmatrix}=0 \qquad (32)$$

After the computation of the determinant we obtain a quadratic equation in *c*:

$$\begin{aligned}&\left(k^4+2\lambda^2k^2\right)c^2+\left[2\beta\left(k^2+\lambda^2\right)-(U_1+U_3)\left(k^4+2\lambda^2k^2\right)\right]c+\\ &\left[k^4U_1U_3+\beta^2-(U_1+U_3)\left(k^2+\lambda^2\right)\beta+\lambda^2k^2\left(U_3^2+U_1^2\right)\right]=0\end{aligned} \qquad (33)$$

where: $\lambda^2=f_0^2/\left(\sigma\Delta p^2\right)$

Solving (32) for the phase speed we obtain:

$$c=U_m-\frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)}\pm\delta^{1/2} \qquad (34)$$

where: $\delta\equiv\frac{\beta^2\lambda^4}{k^4\left(k^2+2\lambda^2\right)^2}-\frac{U_T^2\left(2\lambda^2-k^2\right)}{\left(k^2+2\lambda^2\right)}$ (35)

and: $U_m\equiv\frac{U_1+U_3}{2}$, $U_T\equiv\frac{U_1-U_3}{2}$

U_m and U_T are the vertically averaged zonal wind and the basic state thermal wind for the interval $\Delta p/2$.

We have shown that (28) is a solution for (25)-(27) only if the speed phase satisfies (34).

If $\delta < 0$, the phase speed will have an imaginary part and the perturbations will amplify exponentially, resulting in baroclinic instability.

If $\delta > 0$, we have real roots for the phase speed, resulting in the baroclinic stability.

As an application for this we can consider a barotropic basic state. If we let $U_T = 0$, so that the basic thermal wind vanishes, the phase speeds

$$c_1=U_m-\frac{\beta}{k^2} \qquad (36)$$

and

$$c_2 = U_m - \frac{\beta}{\left(k^2 + 2\lambda^2\right)} \quad (37)$$

are both solutions. These are real quantities. The phase speed c_1 is the dispersion relationship for a barotropic Rossby wave with no y dependence. If we substitute (36) in place of ***c*** in (29)-(31), we will see that in this case A=B and C=0 so that the perturbation is barotropic. The expression (37) can be regarded as the phase speed for an internal baroclinic Rossby wave.

Bibliography

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