can be found by taking derivatives of the appropriate order and evaluating at $\vec{k} = 0$ (only one term will be nonzero).

Another important object is the logarithm of the characteristic function. If we choose to write this as a power series in k of the form (for the 1D case)

$$\log\langle e^{ikx}\rangle = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n \quad , \tag{5.24}$$

this defines the *cumulants* C_n (note that the sum starts at $n = 1$ because $\log 1 = 0$ and so there is no constant term). The cumulants can be found by comparing this to the power series expansion of the characteristic function,

$$\exp\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n\right) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n\rangle \quad , \tag{5.25}$$

expanding the exponential as

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \quad , \tag{5.26}$$

and grouping terms by order of k. The cumulants have an interesting connections to Gaussianity (Problem 5.1).

5.2 STOCHASTIC PROCESSES

It is now time for time to appear in our discussion of random systems. When it does, this becomes the study of *stochastic processes*. We will look at two ways to bring in time: the evolution of probability distributions for variables correlated in time, and stochastic differential equations.

If $x(t)$ is a time-dependent random variable, its Fourier transform

$$X(\nu) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \tag{5.27}$$

is also a random variable but its *power spectral density* $S(\nu)$ is not:

$$\begin{aligned} S(\nu) &= \langle |X(\nu)|^2\rangle = \langle X(\nu)X^*(\nu)\rangle \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \int_{-T/2}^{T/2} e^{-i2\pi\nu t'} x(t')\, dt' \end{aligned} \tag{5.28}$$

(where X^* is the complex conjugate of X, replacing i with $-i$). The inverse Fourier transform of the power spectral density has an interesting form,

$$\begin{aligned} &\int_{-\infty}^{\infty} S(\nu) e^{-i2\pi\nu\tau}\, d\nu \\ &= \int_{-\infty}^{\infty} \langle X(\nu)X^*(\nu)\rangle e^{-i2\pi\nu\tau}\, d\nu \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \int_{-T/2}^{T/2} e^{-i2\pi\nu t'} x(t')\, dt'\, e^{-i2\pi\nu\tau}\, d\nu \end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{i2\pi\nu(t-t'-\tau)}\, d\nu\, x(t)x(t')\, dt\, dt' \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \delta(t-t'-\tau)x(t)x(t')\, dt\, dt' \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)\, dt \\
&= \langle x(t)x(t-\tau)\rangle \quad ,
\end{aligned} \tag{5.29}
$$

found by using the Fourier transform of a *delta function*

$$
\int_{-\infty}^{\infty} e^{-i2\pi\nu t}\delta(t)\, dt = 1 \quad \Rightarrow \quad \delta(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t}\, dt \quad , \tag{5.30}
$$

where the delta function is defined by

$$
\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\, dx = f(x_0) \quad . \tag{5.31}
$$

This is the *Wiener–Khinchin* theorem. It relates the spectrum of a random process to its *autocovariance function*, or, if it is normalized by the variance, the *autocorrelation function* (which features prominently in time series analysis, Chapter 16).

5.2.1 Distribution Evolution Equations

A natural way to describe a stochastic process is in terms of the probability to see a sample value x at a time t (written x_t) given a history of earlier values

$$
p(x_t|x_{t_1}, x_{t_2}, \ldots) \quad . \tag{5.32}
$$

Given starting values for x this determines the probability distribution of the future values. If the distribution depends only on the time differences and not on the absolute time,

$$
p(x_t|x_{t-\tau_1}, x_{t-\tau_2}, \ldots) \quad , \tag{5.33}
$$

then the process is said to be *stationary* (sometimes qualified by calling it *narrow-sense* or *strict* stationarity). A more modest definition asks only that the means and covariances of a process be independent of time, in which case the process is said to be *weak* or *wide-sense* stationary.

If the conditional distribution is limited to a finite history

$$
p(x_t|x_{t-\tau_1}, x_{t-\tau_2}, \ldots, x_{t-\tau_N}) \tag{5.34}
$$

this is said to be an Nth-order *Markov process*. If it depends on just the previous value

$$
p(x_t|x_{t-\tau}) \tag{5.35}
$$

it is simply called a Markov process, and if x and t are discrete variables

$$
p(x_t|x_{t-1}) \tag{5.36}
$$

it becomes a *Markov Chain*. As with ODEs, an Nth-order Markov process for a scalar