Multiple equilibria in the Northern Hemisphere circulation

Extratropical circulation appears to alternate between two states:
1. strong zonal flow and weak waves (high-index state)
2. weak zonal flow and high-amplitude waves (low-index state)

This behavior suggests that there is ***more than one climate regime*** consistent with ***a given external forcing,*** and the observed climate switches back and forth between these regimes in a chaotic fashion.

NOTE: Whether the high-index and low-index states actually correspond to distinct quasi-stable atmospheric climate regimes is a matter of controversy (like many other climatic concepts!!!!)

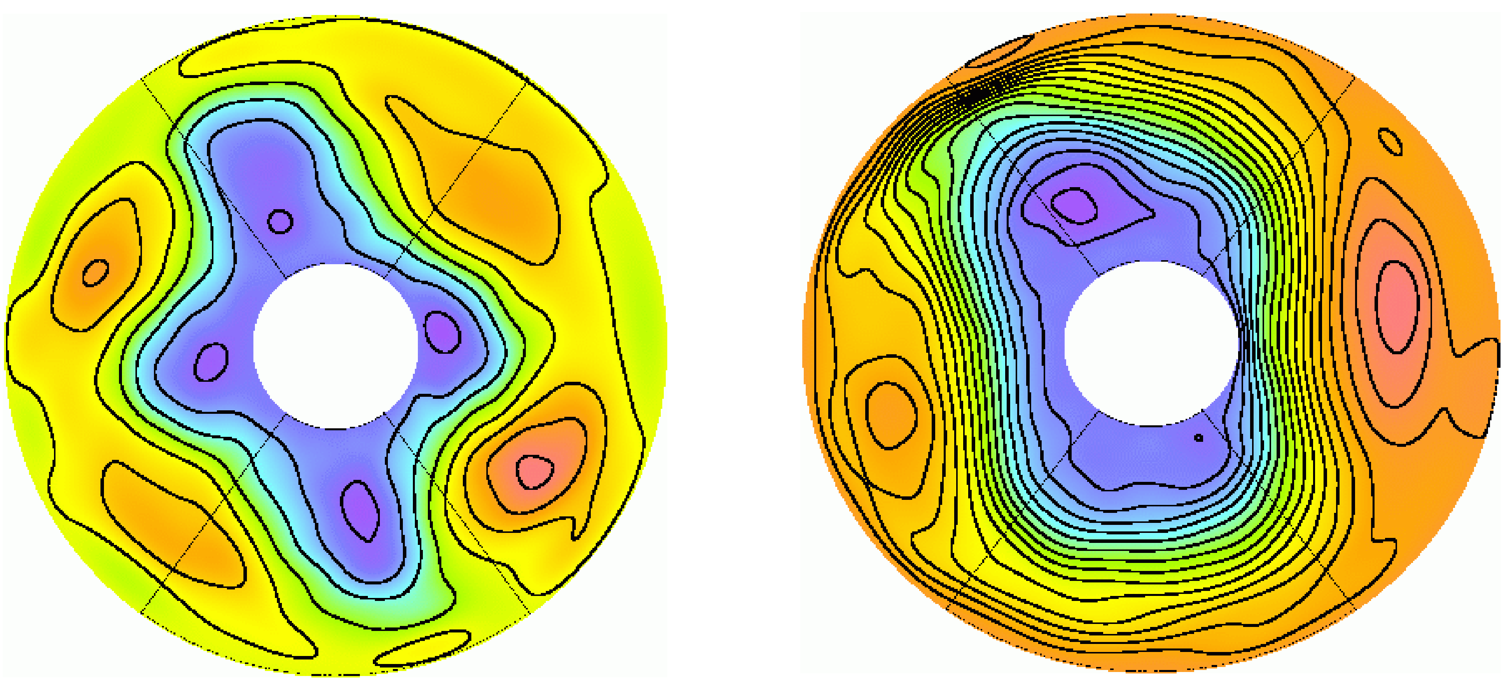


Fig.1. Blocked (low-index) and zonal (high-index) state of the Northern Hemisphere circulation

To understand how these climatic regimes appear, we have to analyze how the Northern Hemisphere atmospheric flow is perturbed by orographic and thermal forcing.

Vorticity equation

1. Cartesian coordinates.

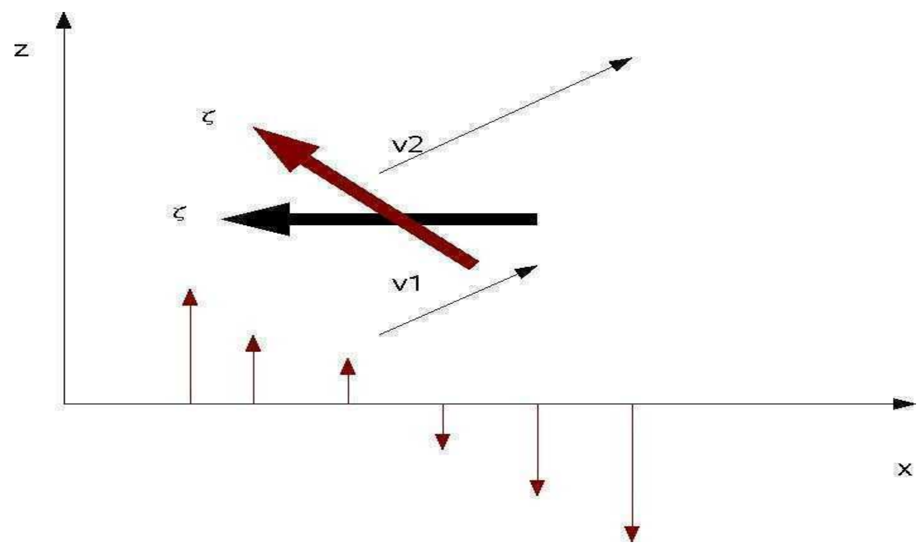
$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y})-(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z})+\frac{1}{\rho^2}(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x})$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

Physical significance of the right-hand side terms:

(1)-divergence term: *if there is a positive horizontal divergence, the area enclosed by a chain of fluid parcels will increase with time,* ***and if circulation is to be conserved,*** *the absolute vorticity of the enclosed fluid must decrease.*

(2)-tilting or twisting term: *vertical vorticity is generated by the tilting of horizontally oriented components of vorticity into the vertical by a non-uniform vertical motion field.*

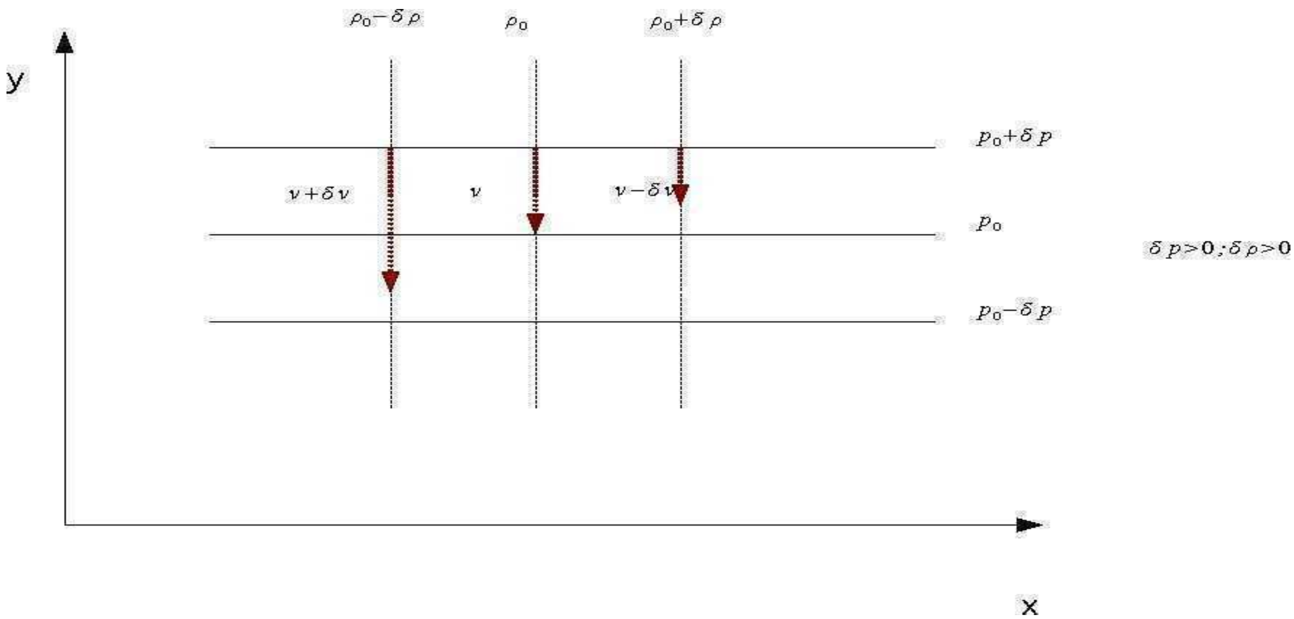


If w=constant and $\frac{\partial v}{\partial z}>0$ *, the vorticity vector is parallel to the x-axis (black arrow). If at the same time* $\frac{\partial w}{\partial x}<0$ *, the vorticity vector is tilted (red arrow) due to advection of horizontal vorticity by the vertical motion.*

Analog, $\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}>0$ *will create a vertical component of vorticity .*

(3)-solenoidal term:

Suppose: $\frac{\partial \rho}{\partial x} > 0$ **and** $\frac{\partial p}{\partial y} > 0$ **that is** $\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} > 0$



The velocity of relative low-density particles is higher than that of relative high-density particles for the same pressure gradient. Therefore, a shear of meridional velocity appears (vertical vorticity is different from zero).

A similar discussion for the $\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$ term.

More general:

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) = -(\nabla \alpha x \nabla p)\vec{k}$$

Therefore, if the angle between the density gradient and the pressure gradient is different from zero, vorticity is generated via solenoidal term.
For synoptic-scale motions the vorticity equation can be simplified using scale analysis. It has the form:

$$\frac{d_h(\zeta + f)}{dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \textbf{(1)}$$

$$\frac{d_h}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

The change of absolute vorticity following the horizontal motion is approximately given by the generation (destruction) of vorticity owing to horizontal convergence (divergence).

BAROTROPIC POTENTIAL VORTICITY EQUATION

If the atmosphere is represented as:
- homogeneous incompressible fluid ($\nabla \vec{v} = 0$)
- variable depth $h(x, y, t) = z_2(x, y, t) - z_1(x, y, t)$ with z1 lower boundary and z2 upper boundary

$$\nabla \vec{v} = 0 \quad \text{imply} \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{-\partial w}{\partial z} \qquad (2)$$

With (2) Equation (1) becomes:

$$\frac{d_h(\zeta + f)}{dt} = (\zeta + f)\left(\frac{\partial w}{\partial z}\right) \qquad (3)$$

Here we retain the relative vorticity in the divergence term for the reason that will become apparent later.
Letting the vorticity in (3) be approximated by the geostrophic vorticity ζ_g and the wind by geostrophic wind (u_g, v_g) , we can integrate vertically from z1 to z2 to get:

$$h\frac{d_h(\zeta + f)}{dt} = (\zeta_g + f)[w(z_2) - w(z_1)] \qquad (4)$$

with w=dz/dt and h=h(x,y,t) and,

$$w(z_2) - w(z_1) = \frac{dz_2}{dt} - \frac{dz_1}{dt} = \frac{d_h h}{dt} \qquad (5)$$

Substituting from (5) in (4), we get: $\dfrac{1}{\zeta_g + f}\dfrac{d_h(\zeta_g + f)}{dt} = \dfrac{1}{h}\dfrac{d_h h}{dt}$

$$\frac{d_h \ln(\zeta_g + f)}{dt} = \frac{d_h \ln h}{dt}$$

which implies:

$$\frac{d_h}{dt}\left(\frac{\zeta_g + f}{h}\right) = 0 \qquad (6)$$

is the potential vorticity conservation theorem for a barotropic fluid. The quantity conserved following the motion in (6) is referred to as **the barotropic potential vorticity.**
If the flow is purely horizontal (w=0), we obtain **the barotropic vorticity equation** :

$$\frac{d_h}{dt}(\zeta_g + f) = 0 \qquad (7)$$

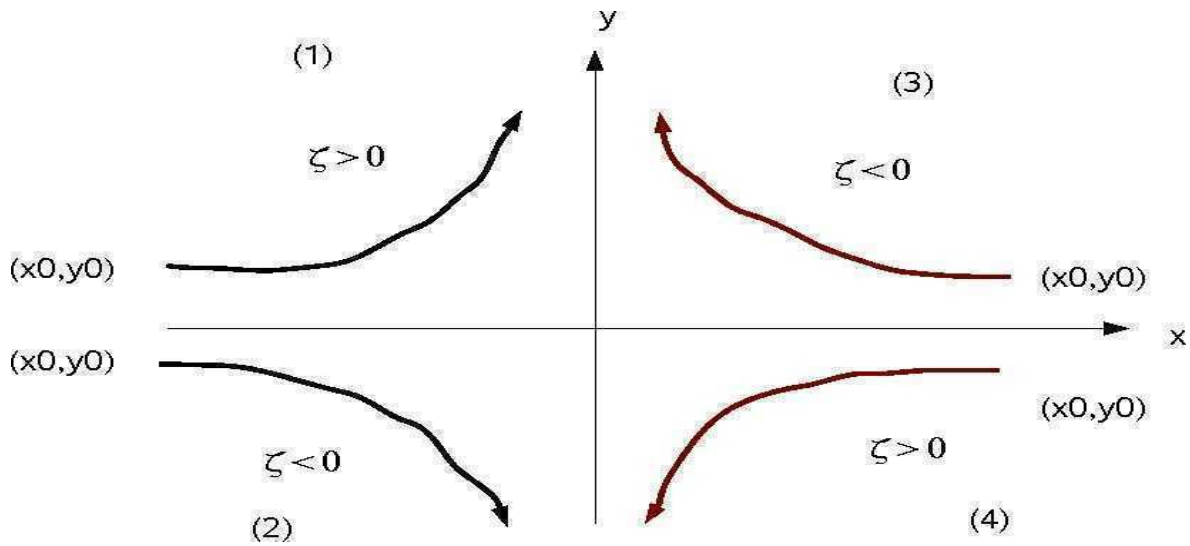
which states that absolute vorticity is conserved following the horizontal motion.

EASTERLY AND WESTERLY FLOWS. CONSERVATION OF ABSOLUTE VORTICITY.

$\eta = \zeta + f$ - absolute vorticity

(x0,y0) - purely zonal flow (relative vorticity is zero)- $\eta(x_0, y_0) = f_0$

Conservation of absolute vorticity implies that at any point the absolute vorticity of a parcel trajectory that passes through (x0,y0) is: $\eta = \zeta + f = f_0$



(1) $\zeta > 0$; f increases; $\zeta + f > f_0$; η not conserved

(2) $\zeta < 0$; f decreases; $\zeta + f < f_0$; η not conserved

(3) $\zeta < 0$; f increases; $\zeta + f = f_0$; η conserved

(4) $\zeta > 0$; f decreases; $\zeta + f = f_0$; η conserved

- for easterly flows (wind blows from east to west) conservation of absolute vorticity is possible both for northward and southward curvature.

- for westerly flows (wind blows from west to east) the motion has to remain purely zonal if the absolute vorticity is conserved.

CONSERVATION OF POTENTIAL VORTICITY. INFLUENCE OF THE TOPOGRAPHY

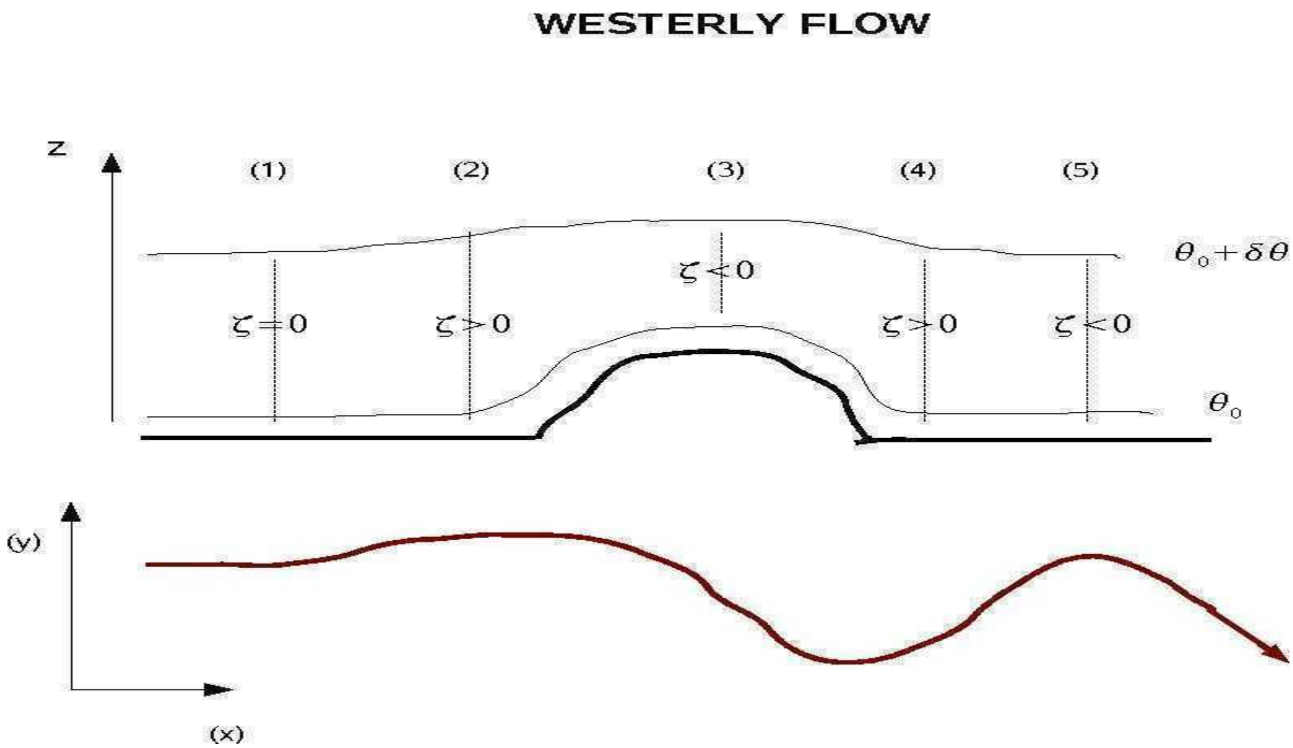
$P=(\zeta_\theta+f)(-g\frac{\partial\theta}{\partial p})=const$ - Ertel potential vorticity in isentropic coordinates

-Potential vorticity is conserved following the motion in adiabatic frictionless flow

Let a large-scale flow of air over a large mountain barrier in which $\partial\theta/\partial p$ undergoes a substantial change along the trajectory.

Westerly flow

- isentropic surface θ_0 follows approx. the topography
- $\theta_0+\delta\theta$ isentropic surface is deflected vertically, but the vertical displacement at upper levels is spread horizontally; it extends upstream and downstream of the barrier and has a smaller amplitude in the vertical



(1) zonal flow ($\zeta=0$)

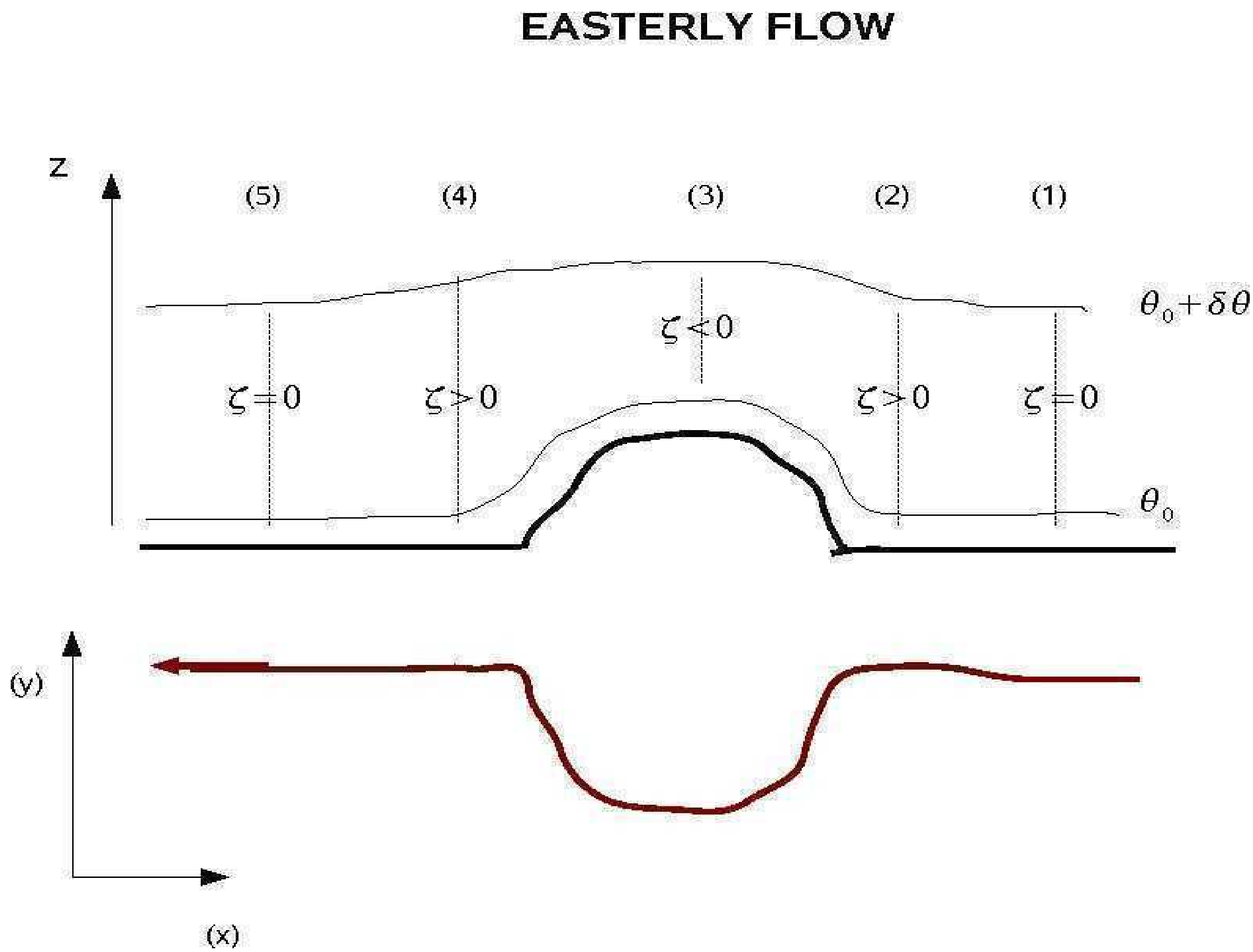
(2) $\zeta>0$ (vertical stretching of air columns upstream of the topographic barrier); $-\partial\theta/\partial p$ decreases; ζ must become positive to conserve potential vorticity; air column turns cyclonically as it approaches the topographic barrier; poleward drift-f increases)

(3) $\zeta<0$ (vertical extent of the air parcel decreases; relative vorticity must become negative; anticyclonic vorticity and southward displacement)

(4) $\zeta>0$ (air column has passed over the mountain and returns to original depth; is located south of original latitude; f is smaller and relative vorticity must be positive; the trajectory must have cyclonic curvature)

(5) $\zeta<0$ (the parcel returns to its original latitude; still has a poleward velocity component and will continue poleward, gradually acquiring anticyclonic curvature until its direction is again reversed)

EASTERLY FLOW



(1) $\zeta = 0$ (zonal flow)

(2) $\zeta > 0$ (vertical stretching; cyclonic flow; equatorward component of motion; f decreases)

(3) $\zeta < 0$ (vertical contraction; decrease of absolute vorticity due to both anticyclonic relative vorticity and a decrease in f owing to the equatorward motion)

(4) $\zeta > 0$ (the same depth as in (2))

(5) $\zeta = 0$ (purely zonal flow)

Conclusion:
The dependence of the Coriolis parameter on latitude creates a dramatic difference between westerly and easterly flows over large-scale topographic barriers. For westerly flows, topographic barriers generate wavelike disturbances in the streamlines that extend far downstream. For easterly flows, the disturbance in the streamlines damps out away from the barrier.

FREE BAROTROPIC ROSSBY WAVES

The barotropic vorticity equation (7) states that the vertical component of absolute vorticity is conserved following the horizontal motion. For mid-latitude beta plane it has the form:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\zeta+\beta v=0 \qquad (8)$$

Assuming that the motion consists of a basic state zonal velocity plus a small horizontal perturbation:

$u=\bar{u}+u'$ $\quad v=v'$ and $\quad \zeta=\bar{\zeta}+\zeta'$

the solution of eq. (8) is the **'free batrotropic Rossby wave'.** Such a wave has the phase speed c_x :

$$c_x-\bar{u}=-\frac{\beta}{K^2} \qquad (9)$$

with

$K^2=k^2+l^2$ the total horizontal wave number squared.
The free Rossby wave zonal phase propagation is always **westward** relative to the mean zonal flow.

Rossby waves are dispersive waves, that is their phase speed depends on the wave number (speed increases with increasing wavelength). For longer wavelengths, the Rossby wave phase speed may be large enough to balance the eastward advection by the mean zonal wind, so that the resulting disturbance is stationary relative to the surface of the earth. For this case, from (9) we obtain:

$$K^2=\frac{\beta}{\bar{u}}=K_s^2 \qquad (10)$$

which is referred to as **resonant stationary Rossby wave number**.

For a typical mid-latitude synoptic scale disturbance, with a similar meridional and zonal scale ($l\approx k$) and a zonal wavelength of the order 6000 km, the Rossby wave speed relative to the zonal flow calculated with (9) is approximately -8m/s. Because zonal wind is usually greater than 8m/s, synoptic-scale Rossby waves usually move eastward.

FORCED TOPOGRAPHIC ROSSBY WAVES

Under the assumption that the upper boundary is at fixed height H, and the lower boundary is at variable height $h_T(x, y)$ as well as that $|\zeta_g| \ll f_0$ (quasigeostrophic approximation) equations (4) and (5) become:

$$H(\frac{\partial}{dt} + \vec{v}\nabla)(\zeta_g + f) = \frac{-f_0}{H}\frac{d\,h_T}{dt} \qquad (11)$$

After linearizing and applying the midlatitude beta-plane approximation (11) becomes:

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\zeta'_g + \beta v'_g = \frac{-f_0}{H}\bar{u}\frac{\partial h_T}{\partial x} \qquad (12)$$

We look for the solution of (12) for the special case of a sinusoidal lower boundary:

$$h_T(x, y) = \Re[h_0 \exp(ikx)]\cos(ly) \qquad (13)$$

and streamfunction:

$$\psi(x, y) = \Re[\psi_0 \exp(ikx)]\cos(ly) \qquad (14)$$

With (13) and (14) and (12) we get:

$$\psi_0 = \frac{f_0 h_0}{H(K^2 - K_s^2)} \qquad (15)$$

The streamfunction is exactly in phase (ridges over the mountains) or exactly out of phase (troughs over the mountains) with the topography depending on the sign of $K^2 - K_S^2$. For long waves (K<Ks), the topographic vorticity source in (12) is primarily balanced by meridional advection of planetary vorticity (the beta effect). For short waves (K>Ks), the source is balanced primarily by the zonal advection of relative vorticity.

For K=Ks, we have a singularity (infinite amplitude!!). To remove this singularity, we include a damping term of vorticity due to boundary layer drag in the form of Ekman pumping. The vorticity equation takes the form:

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\zeta'_g + \beta v'_g + r\zeta'_g = \frac{-f_0}{H}\bar{u}\frac{\partial h_T}{\partial x} \qquad (16)$$

where $r = \tau_\epsilon^{-1}$ is the inverse of the spin-down time.

For steady flow, (16) has a solution with complex amplitude:

$$\psi_0 = \frac{f_0 h_0}{H(K^2 - K_s^2 - i\epsilon)} \qquad (18)$$

$$\epsilon = \frac{rK^2}{k\bar{u}}$$

Thus, the boundary layer drag shifts the phase of the response and removes the singularity at resonance. The amplitude is maximum at K=Ks and the trough in the streamfunction occurs ¼ cycle east of the mountain crest, like in observations.

CLIMATE REGIMES

We examine the equilibrium mean states that can result when a damped topographic Rossby wave interacts with zonal mean flow. **(Charney, J.G. and J.G. De Vore (1979): Multiple flow equilibria in the atmosphere and blocking, JAS).** In this model the wave disturbance is governed by :

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\zeta'_g+\beta v'_g+r\zeta'_g=\frac{-f_0}{H}\bar{u}\frac{\partial h_T}{\partial x}$$

which is a linearized form of the barotropic vorticity equation

$$H\left(\frac{\partial}{dt}+\vec{v}\nabla\right)(\zeta_g+f)=\frac{-f_0}{H}\frac{dh_T}{dt}$$

with weak damping added. We showed that for:

$$h_T(x,y)=\Re[h_0\exp(ikx)]\cos(ly)$$

$$\psi(x,y)=\Re[\psi_0\exp(ikx)]\cos(ly)$$

we have:

$$\psi_0=\frac{f_0 h_0}{H(K^2-K_s^2-i\epsilon)}$$

with $\epsilon=\frac{rK^2}{k\bar{u}}$

Assume that the zonal mean (OVERBAR) flow is governed by the barotropic momentum equation:

$$\frac{\partial\bar{u}}{\partial t}=-D(\bar{u})-k(\bar{u}-U_e)$$

The first term is the forcing term due to the interaction of wave and mean flow. The second term is a linear relaxation toward an externally determined basic state flow, U_e . With the above relations, it is possible to show that the forcing term has the form:

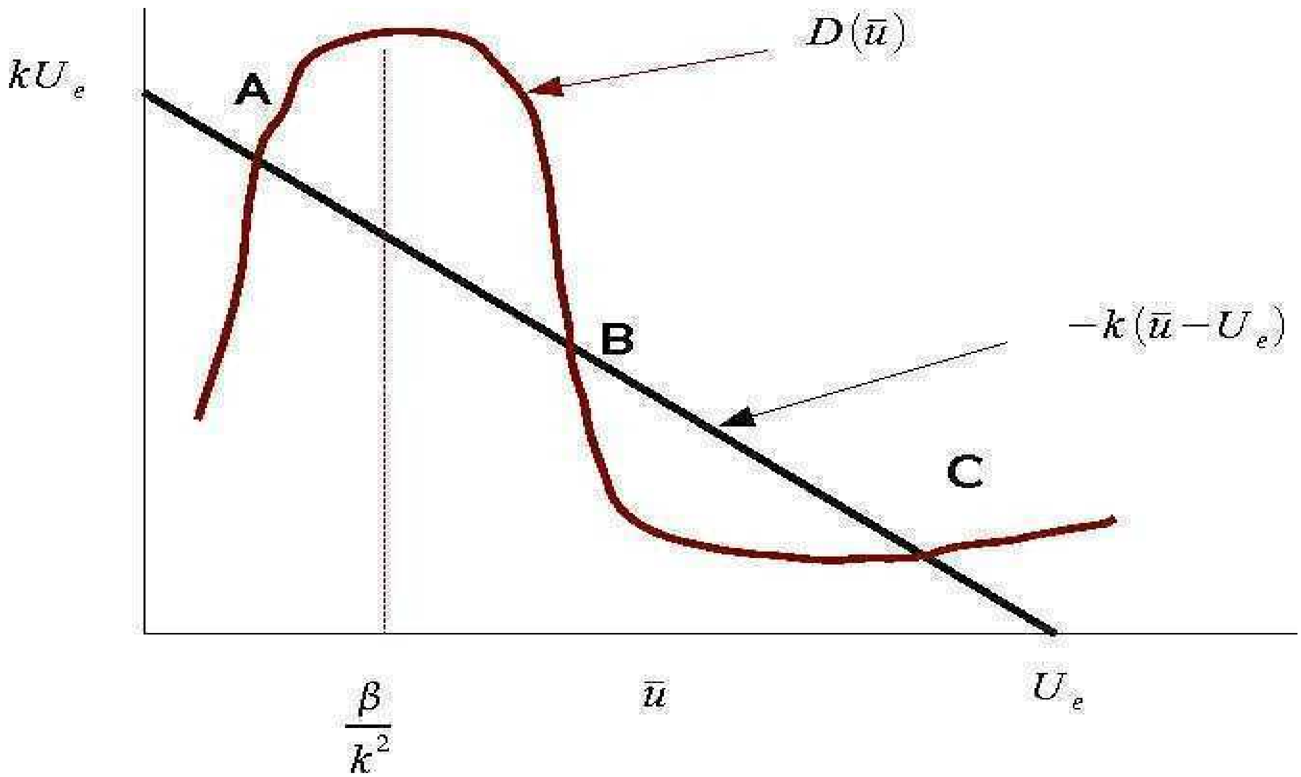
$$D(\bar{u})=-\overline{v'_g\zeta'_g}-\frac{f_0}{H}\overline{v'_g h_T}$$

If h_T and the eddy geostrophic streamfunction are assumed to consist of a single harmonic wave component in x and y respectively (above relations), the vorticity flux vanishes, and the forcing term can be expressed as:

$$D(\bar{u})=-\left(\frac{f_o}{H}\right)\overline{v'_g h_T}=\left(\frac{rK^2 f_0^2}{2\bar{u}H^2}\right)\frac{h_0^2\cos^2 ly}{(K^2-K_s^2)+\epsilon^2} \qquad (22)$$

with Ks being the resonant stationary Rossby wave number.

Solutions of the Charney-DeVore model (Holton, 1992).



-A stable solution corresponding to a **low-index equilibrium** with high-amplitude waves analogous to a blocking regime
-C stable solution corresponding to a **high-index equilibrium** with low-amplitude waves analogous to a zonal regime
-B unstable solution
In **THIS MODEL,** there are two possible „climates“ associated to the same forcing.

Models that contain more degrees of freedom (GCMs for example!) do not generally have multiple steady solutions. There is a tendency for the (unsteady) solutions to cluster about certain climate regimes and to shift between regimes in an unpredictable fashion. The irregularity of the flow dynamics generated by these **DETERMINISTIC MODELS** is characteristic of a wide range of nonlinear dynamical systems and is referred to as **DETERMINISTIC CHAOS.**

LOGISTIC MAP

The logistic equation (sometimes called the Verhulst model or logistic growth curve) is a model of population growth first published by Pierre Verhulst (1845, 1847). The model is continuous in time:

$$\frac{dx}{dt}=ax(1-x)$$

a - control parameter

A modification of the continuous equation to a discrete quadratic recurrence equation known as the logistic map:

$x_{n+1}=f(x_n),\ n=0,1,2,3....$ with $f(x)=ax(1-x)$

If the control parameter varies in the [0,4] interval and x in [0,1] interval, the values of f(x) varies in [0,1] interval.

Solutions of the equation: $f(x)=x$ -----fixed points (stable or unstable)

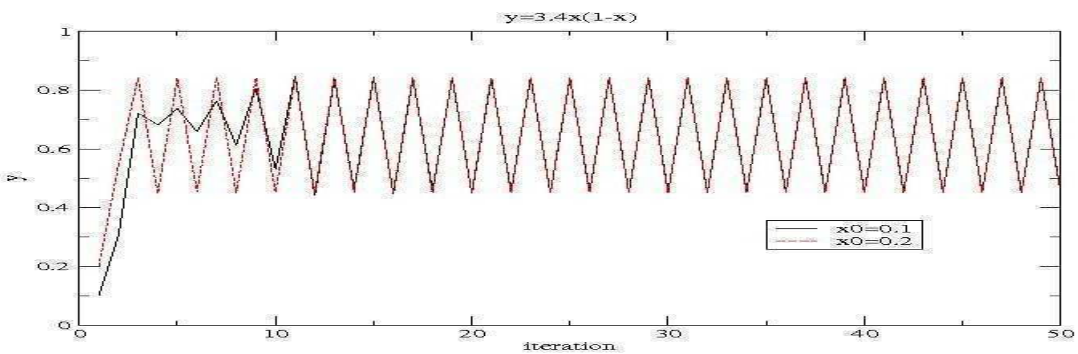
Solutions of the equation: $f^2(x)=f(f(x))=x$ ---periodic points of period two (stable or unstable) and the sequence: x, f(x) is referred to as 2-cycle

Solutions of the equation: $f^n(x)=x$ -periodic points of period n and $x, f(x) .... f^n(x)$ n-cycle

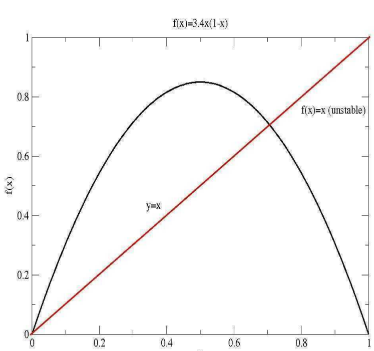
We investigate two particular cases**: a=3.4 and a=4.**

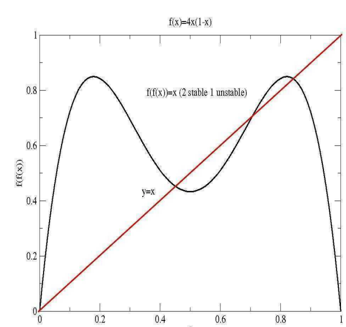
1)a=3.4

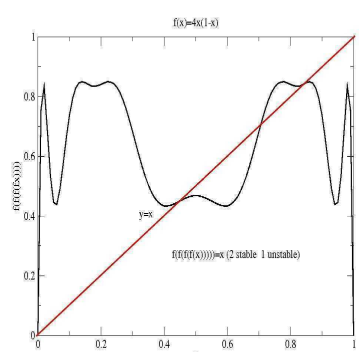
Two orbits starting from x0=0.1 and x0=0.2 are attracted by a stable 2-cycle. (***No sensible dependence on initial conditions!!)***



The map has one **unstable (repellor) fixed point** and **one stable (attractor) 2-cycle .**

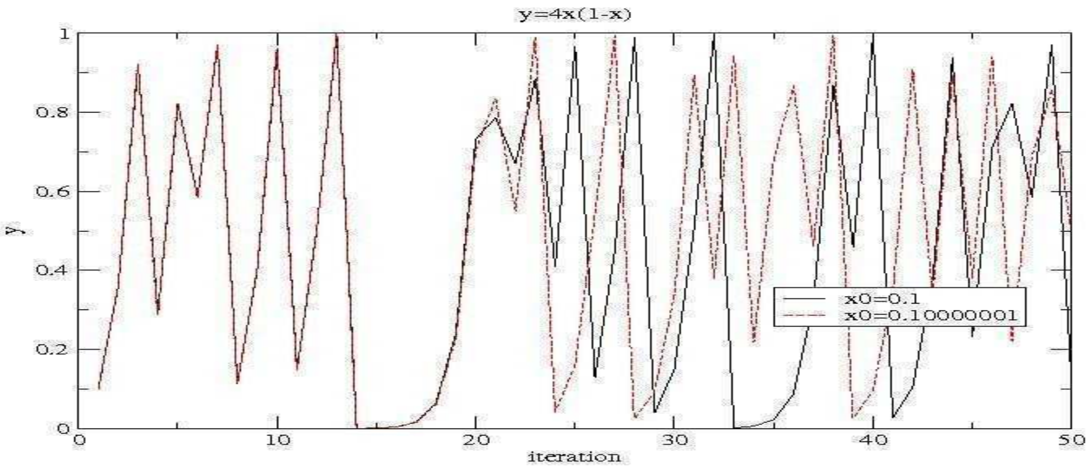




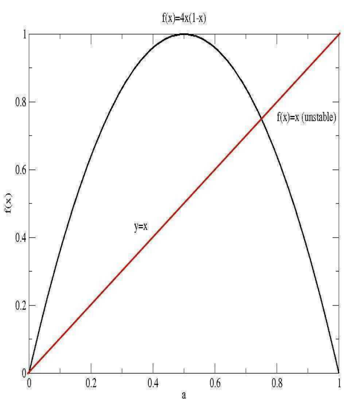


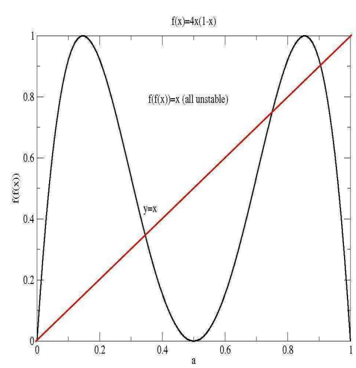
All trajectories starting from [0,1] interval are attracted by this cycle.

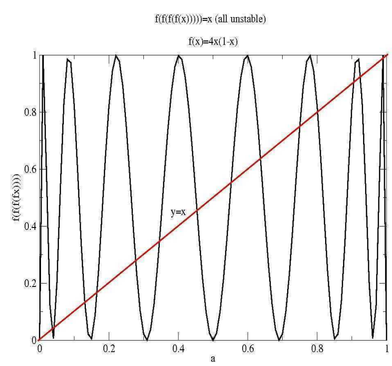
2)a=4.0



*Two trajectories starting from very close initial conditions (x0=0.1 and x0=0.0000001) become uncorrelated after several iterations. For this value of the control parameter the logistic map has **'sensible dependence on initial conditions'***

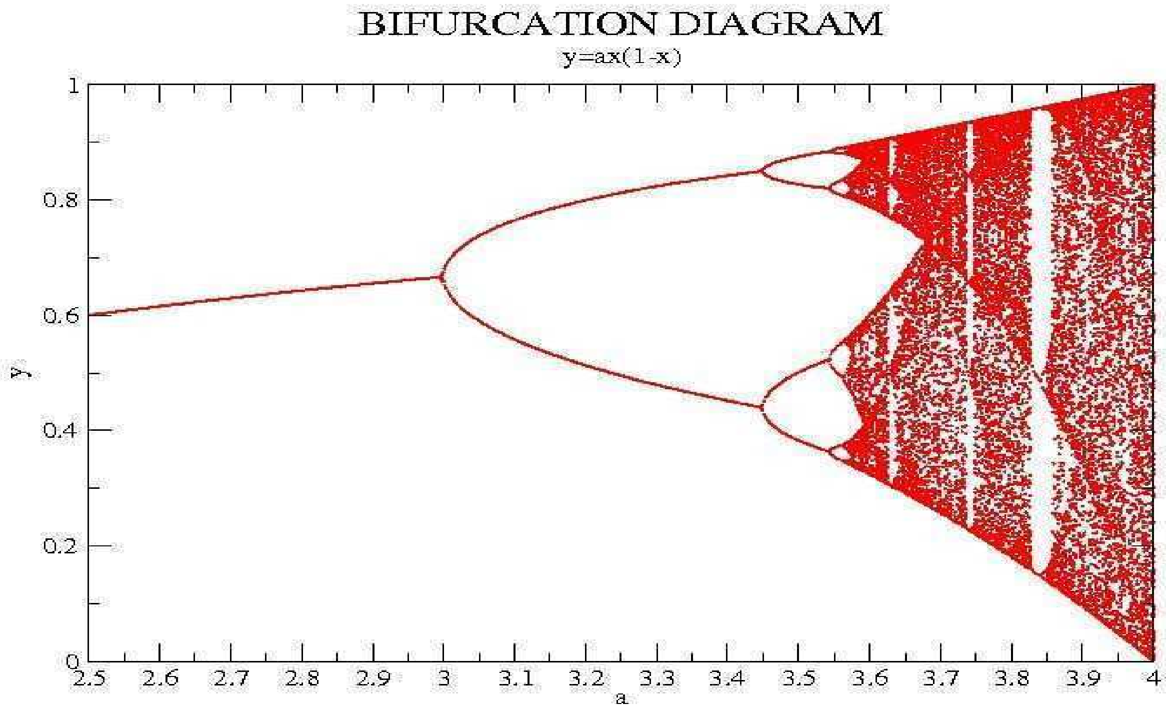






*All fixed and periodic points are **unstable (repellors).** For n going to infinity the logistic map has an infinite number of unstable fixed and periodic points. The orbits starting from different points in the [0,1] interval are uncorrelated.*
*The logistic map can be used to generate **random numbers**.*

BIFURCATION DIAGRAM



How is the above plot obtained?

1. Choose an initial condition (x0=0.1).
2. Choose the parameter control interval.
3. For each value of control parameter calculate the first n (say n=50!!) points of the orbit that starts from x0 using the logistic map.
Plot the next n1 points of the orbit (say n1=100!) versus the value of control parameter .

In other words, the set of fixed points of f^n ***corresponding to a given value of a are plotted for values for an increase to the right.***

The ***Feigenbaum constant*** $\delta = 4.669201609102990671853203382\cdots$ *is the limiting ratio between successive bifurcation intervals.*

The logistic map goes from order to chaos through a cascade of ***period-doubling bifurcations****. Other routes from order to chaos: successive Hopf bifurcations, intermittency, etc.*

LORENZ SYSTEM

Model of a convective flow (E. Lorenz, Deterministic nonperiodic flow, JAS, 1963).
Equations:

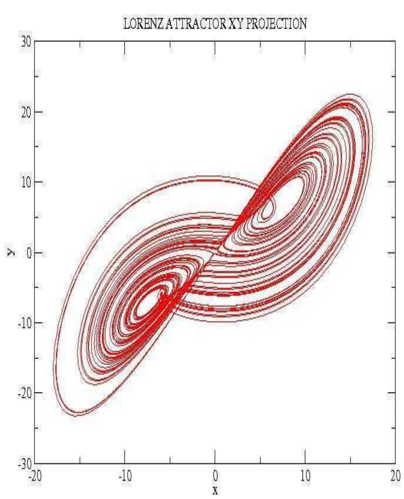
$$\dot{x} = s(y - x) = f_1(x, y)$$
$$\dot{y} = rx - y - xz = f_2(x, y, z)$$
$$\dot{z} = xy - bz = f_3(x, y, z)$$

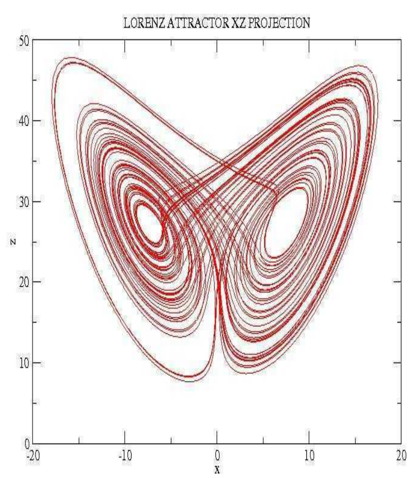
***s** the "Prandtl number." (the ratio of the fluid viscosity of a substance to its thermal conductivity).*
***r** the difference in temperature between the top and bottom of the fluid system.*
***b** the width to height ratio of the box which is being used to hold the fluid.*

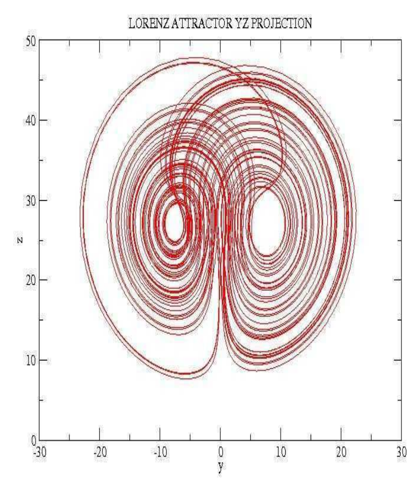
We numerically solve the system using the following discretization scheme:

$$x(t+dt) = x(t) + f_1(x, y) * dt$$
$$y(t+dt) = y(t) + f_2(x, y, z) * dt$$
$$z(t+dt) = z(t) + f_3(x, y, z) * dt$$

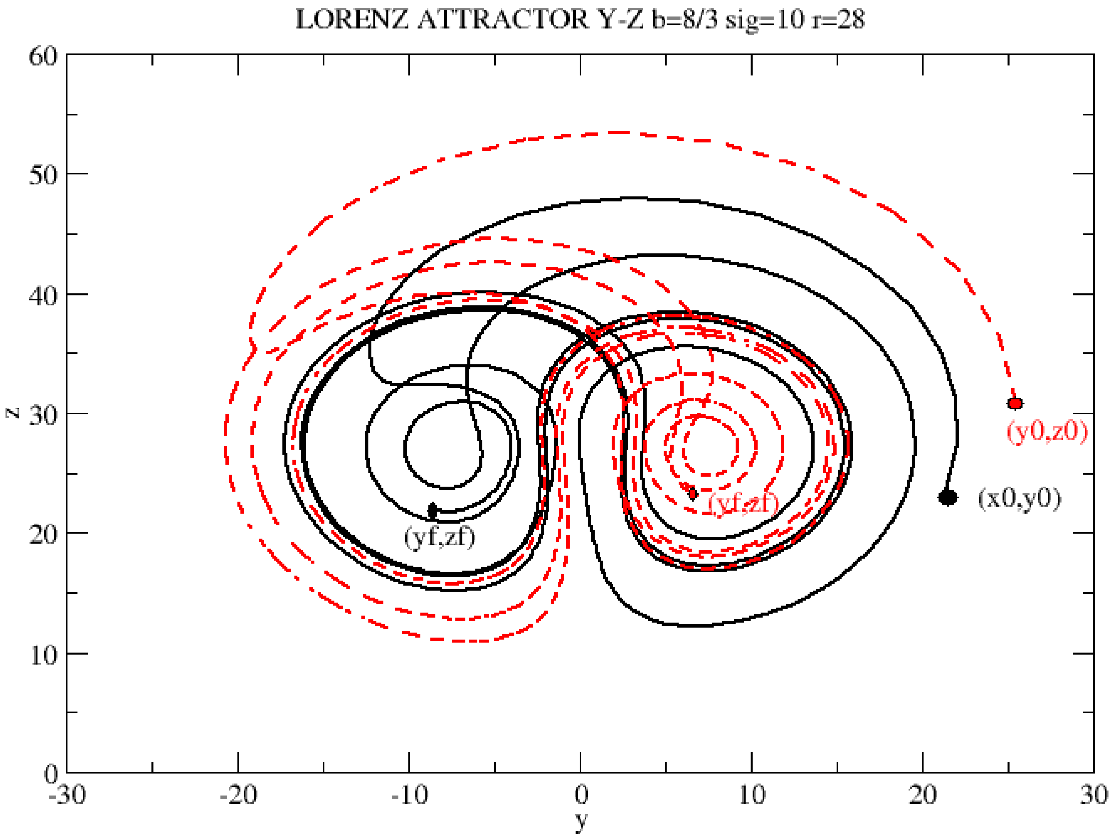
For **s=10, b=8/3 and r=28** (Lorenz initial values), dt=0.001 and initial conditions x0=0.1, y0=0.1 , z0=0.1, we obtain the Lorenz attractor (pictures below!!).







THE BUTTERFLY EFFECT



*Two initial close trajectories (red and blue), after a transitory regime, move independently on the attractor (**sensible dependence on initial conditions),** like in the case of the logistic map.*

*Sensitive dependence on initial conditions is sometimes referred to as the **butterfly effect***

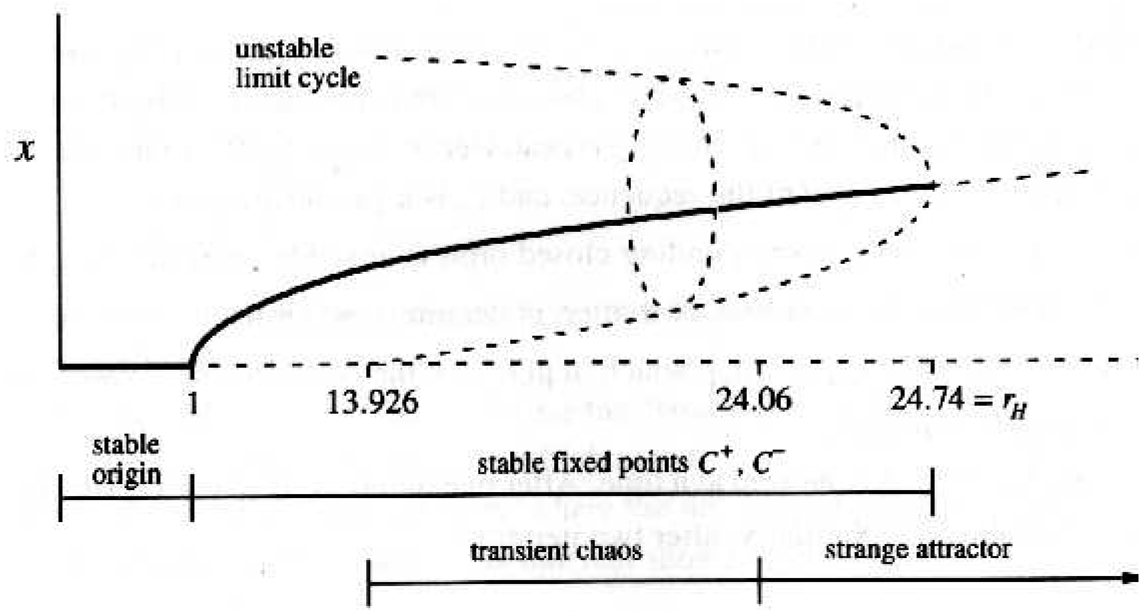
The butterfly effect refers to the idea that a butterfly's wings might create tiny changes in the atmosphere that ultimately cause a tornado to appear (or, for that matter, prevent a tornado from appearing). The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale phenomena. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different.

BIFURCATION DIAGRAM

With **s=10, b=8/3** and **r variable** (as a control parameter) the system will exhibit the following route to chaos:

1. $0 < r < 1$ the origin (0,0,0) is a globally **stable fixed point.** At r=1 this fixed point becomes unstable via a supercritical pitchfork bifurcation, and two new stable fixed points appear: [$+\sqrt{b(r-1)}$, $+\sqrt{b(r-1)}$, $r-1$] and [$-\sqrt{b(r-1)}$ $-\sqrt{b(r-1)}$ $r-1$].

2.At **r=24.74** the fixed points lose stability in a subcritical Hopf bifurcation. When decreasing r, the limit cycles get larger until the cycles touch at r=13.926. Below r=13.926 there are no limit cycles. Above r=13.926 the system exibits *tdansient chaos.*



Bifurcation diagram as derived from numerical integration of Lorenz system (below!!)