Dynamics I: Exam

Time: Monday, 26. Febr. 2007, 10:00-11:45, Room H1, 120 points, *100 % = 100 points*
Devices: one (!) sheet of paper which has to be delivered at the end; calculator (if you want).

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1) Several questions about the course (*20 points*).

Q1: On a spherical earth (or planet or star, in general), the Coriolis parameter f is proportional to the rotation rate Ω times the sine of the latitude φ :

$$f = 2\Omega \sin\varphi$$

Consider the coordinate y (oriented northward and measured from a reference latitude φ_0), then

$$\varphi = \varphi_0 + y/a$$

with a as the Earth radius (6371 km). Assume y/a as a small departure, the Coriolis parameter can be expanded in a Taylor series with first order in y/a :

2 points

Q2: Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force? What about the Southern Hemisphere? Could you test this in a bathtube with typical size 2m x 1m?

3 points

Q3: Please write down the barotropic potential vorticity equation for large-scale motion!

3 points

Q4: What are the dominant terms in the momentum balance for the large-scale dynamics at mid-latitudes?

3 points

Q5: What is the necessary condition for stability in a linear system

$$\frac{d}{dt}x = Ax$$

with real vector x and $N \times N$ matrix A? ... and for the analogous non-linear case $\frac{d}{dt}x = f(x)$?

3 points

Q6: Please write down the budgets for mass as well as volume conservation! Under which conditions they are equivalent?

3 points

Q7: Please write down diffusion and advection equation for the variables temperature T and three-dimensional velocity v!

3 points

2) Rossby, gravity, and Kelvin waves

Start with the shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \quad (2)$$

$$\frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \quad (3)$$

with H=const. as mean depth and η as surface anomaly.

a) with the elimination of the fast gravity waves in ()

$$\frac{\partial \eta}{\partial t} = 0$$

derive the dispersion relation for divergence-free Rossby waves! Ansatz: Introduce a streamfunction for u,v:

$$\Psi \sim \exp(ikx + ily - i\omega t)$$

b) with the assumption of $f = f_0 = 0$ derive the dispersion relation for gravity waves! The restoring force is related to gravity. Ansatz: take one of the equations (1,2,3) and derive the solution.

c) Inertia-Gravity wave (Poincare wave). As in b), but now $f = f_0 \neq 0$. Ansatz:

$$(u, v, \eta) = (\hat{u}, \hat{v}, \hat{\eta}) \times \exp(ikx + ily - i\omega t)$$

and derive algebraic equations for $(\hat{u}, \hat{v}, \hat{\eta})$. From that, one can derive the dispersion relation from the condition $detA = 0$ with the 3×3 matrix A originating from the algebraic equations $A(\hat{u}, \hat{v}, \hat{\eta})^T = 0$. The trivial solution $\omega = 0$ can be related to a steady flow. For the non-trivial solution, the frequency ω is as in b) for $f = 0$ (classical gravity waves), and $\omega = f$ for low wave numbers $k, l \rightarrow 0$ (inertial oscillation, all fluid particles move in unison).

d) Kelvin wave. Assume a vertical wall at x=0 along the y-axis (an idealized coast) and u=0. Derive the solution for $v(x, y, t)$ and $\eta(x, y, t)$ using the equations (2,3)! Specify the x-dependence of the solutions using (1) and discuss the trapping distance from the coast!

30 points

3) Rossby wave formula (long waves in the westerlies)

Consider the vorticity equation

$$\frac{D}{Dt}[(\zeta + f)/h] = 0 \tag{4}$$

with h =const., u and v are the velocity components.

a) Assume a mean flow with constant zonal velocity U

$$u = U = const > 0 \tag{5}$$

and a varying north-south component

$$v = v(x, t) \tag{6}$$

which gives the total motion a wave-like form. Derive the vorticity equation!

b) With the ansatz

$$v(x, t) = A \cos[(kx - \omega t)] \tag{7}$$

determine the disperion relation $\omega(k)$, group velocity $\frac{\partial \omega}{\partial k}$, and the phase velocity $c = \omega/k$.

c) Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.

d) A typical wavelength is 6000 km, a typical U is 15 m/s. Does the wave propagate from east to west or opposite?

30 points

4) Advection and radiation in the atmosphere:

The temperature at a point 360 km north of a station is 5°C lower that at the station. If the wind is blowing from northeast with $v = \sqrt{2} \times 10$m/s and the air is being heated by radiation at a rate of 1°C/h, what is the local temperature change at the station? [Use: 1 h = 3600 s]

Initially, the temperature was 12°C, what is the local temperature after 2 hours?

10 points

5) Potential vorticity:

An air column at 53°N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km hight at 30°N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

Assume: $\sin 53° = 0.8; \sin 30° = 0.5$

The angular velocity of the Earth $\Omega = 2\pi/(1day)$.

10 points

6) Ocean thermohaline circulation: Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{8}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{9}$$

with pressure $p(x, y, z, t)$.
Use the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho \tag{10}$$

and equation (8) in order to derive the meridional overturning stream function $\Phi(y, z)$ as a fuction of density ρ at the basin boundaries! Φ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}}\, d\tilde{z} \tag{11}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z})\, dx \quad \text{(zonally integrated transport),} \tag{12}$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.
Consider now a water planet with flat bottom (unlike the Earth). What is the meridional overturning stream function $\Phi(y, z)$ in this ocean ?

10 points

7) Bifurcation:
Consider the differential equation

$$\frac{d}{dt}x = ax - x^2 \tag{13}$$

Similar dynamics can be derived from the logistic growth or Stommel's box model of the ocean circulation (see lecture). Calculate the (transcritical) bifurcation with respect to the parameter a and draw the bifurcation diagram!

10 points