Dynamics I - 2/11 Lecture Minutes

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1 Advection

The advection equation in 1-dimension is:

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x}$$

With:

$$c(x,t)\ ;\ f(x) = c(x, t = 0)$$

The solution to this equation is

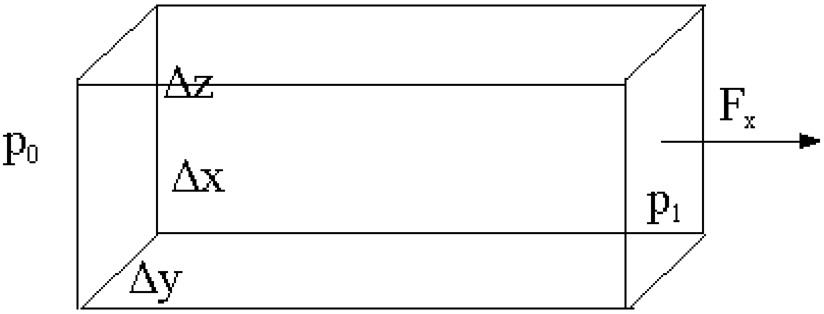
$$c(x,t) = f(x - ut)$$

As you can see by the fact that the derivatives of $f(x - ut)$ on both sides of the equation give the same result.

This represents transport of concentrations in fluids, as the shape of the function f doesn't change with time, but move along the axis x.

2 Explanation of terms in the equation of motion (Navier-Stokes equations)

2.1 Pressure forces



On a volume of fluid with dimensions $\Delta x \Delta y \Delta z$ with pressure $p(x=0) = p_0$ and $p(x = \Delta x) = p_1$the force in the x direction will be

$$F_x = (p_0 - p_1)\Delta y \Delta z$$

that we can approximate as

$$F_x = -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z$$

If we take $f_x = \frac{F_x}{\Delta m}$(specific force or acceleration):

$$f_x = -\frac{\partial p}{\partial x}\frac{\Delta x \Delta y \Delta z}{\Delta m} = -\frac{\partial p}{\partial x}\frac{\Delta V}{\Delta m} = -\frac{\partial p}{\partial x}\frac{1}{\rho}$$

where $\rho$is the density, mass per unit volume.

2.2 Gravitation forces

By Newton's Law on a body whose center of mass is at a distance r from the center of mass of the Earth is applied a force $\vec{F_g}$ by the gravitational field of the Earth itself:

$$\vec{F_g} = -\frac{GMm}{r^3}\vec{r}$$

where M is the Earth's mass, G is the gravitational constant, m the mass of the body.

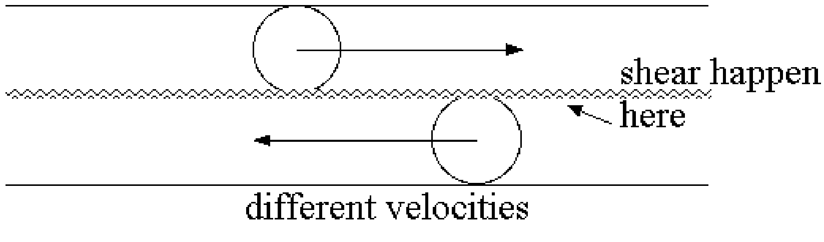
If we take a as the Earth's radius and z as the vertical coordinate from the surface we'll have for the specific force $\vec{f_g}$:

$$\vec{f_g} = -\frac{GM}{(\vec{a}+\vec{z})^2}\frac{\vec{r}}{r} = -g_0\frac{a^2}{(a+z)^2}\frac{\vec{r}}{r} = -g_e\frac{\vec{r}}{r}$$

where we define $g_0 = \frac{GM}{a^2}$ and $g_e = g_0\frac{a^2}{(a+z)^2}$. But since $a \approx$6400 km and an average $z =$10 km, we see that $g_e \approx 1$.

2.3 Friction forces

Friction forces in geophysical fluids are mainly due to *shear* effects, where two fluid layers at different velocity touch and there is a mutual drag:



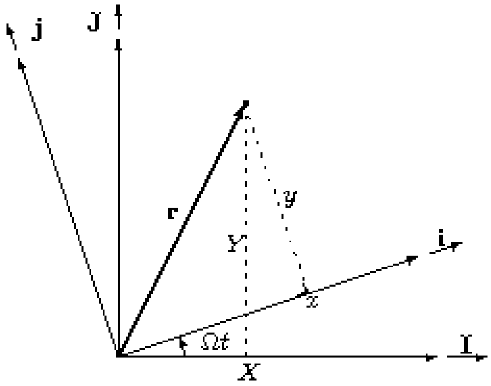
This can be expressed as a force:

$$\vec{f_v} = \frac{1}{\rho}[\nabla \cdot (\mu \nabla \vec{v}) + \nabla(\lambda \nabla \cdot \vec{v})]$$

$\lambda$and $\mu$are experimental friction parameters, while with the usual notation:
$\nabla \cdot \vec{q}$ is the divergence of vector q
∇s is the gradient of scalar s
$\nabla \vec{q}$ is a vector whose components are the gradients of the scalar components of vector q:

$$(\nabla \vec{q})_i = \nabla q_i = (\frac{\partial q_i}{\partial x_1} + ... + \frac{\partial q_i}{\partial x_n})$$

2.4 Coriolis forces



When a coordinate frame is rotating with respect to an inertial frame of reference, the accelerations in the rotating frame can be regarded as apparent forces. In the picture above (X, Y) is the inertial frame, with orthonormal vector basis $E = (\hat{I}, \hat{J})$, and (x, y) is the frame rotating with angular velocity Ω, with basis $e = (\hat{\imath}, \hat{\jmath})$. The relations between the two bases are:

$$\begin{aligned} \hat{\imath} &= \hat{I}\cos\Omega t + \hat{J}\sin\Omega t \\ \hat{\jmath} &= -\hat{I}\sin\Omega t + \hat{J}\cos\Omega t \end{aligned}$$

that is $e = RE$, where R is the rotation matrix:

$$R = \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix}$$

From $E = R^{-1}e$ we have:

$$\begin{aligned} \hat{I} &= \hat{\imath}\cos\Omega t - \hat{\jmath}\sin\Omega t \\ \hat{J} &= \hat{\imath}\sin\Omega t + \hat{\jmath}\cos\Omega t \end{aligned}$$

And for every point (x, y)we have:

$$\begin{aligned} x &= X\cos\Omega t + Y\sin\Omega t \\ y &= -X\sin\Omega t + Y\cos\Omega t \end{aligned}$$

We define now the velocity vector $\vec{u} = (u, v)$ in the (x, y) frame as:

$$\vec{u} = (\frac{dx}{dt}, \frac{dy}{dt}) = (\dot{x}, \dot{y})$$

and in the same way:

$$\vec{U} = \dot{X}\hat{I} + \dot{Y}\hat{J} = (\dot{X}\cos\Omega t + \dot{Y}\sin\Omega t)\hat{\imath} + (-\dot{X}\sin\Omega t + \dot{Y}\cos\Omega t)\hat{\jmath}$$

$$\begin{aligned} u &= \dot{X}\cos\Omega t + \dot{Y}\sin\Omega t - \Omega X\sin\Omega t + \Omega Y\cos\Omega t = U + \Omega y \\ v &= -\dot{X}\sin\Omega t + \dot{Y}\cos\Omega t - \Omega X\cos\Omega t - \Omega Y\sin\Omega t = V - \Omega x \end{aligned}$$

Now we take the second derivative $\ddot{x} = \dot{u}$:

$$\begin{aligned} \ddot{x} &= \ddot{X}\cos\Omega t + \ddot{Y}\sin\Omega t + \frac{d}{dt}(\Omega y) - \Omega\dot{X}\sin\Omega t + \Omega\dot{Y}\cos\Omega t \\ \ddot{y} &= -\ddot{X}\sin\Omega t + \ddot{Y}\cos\Omega t - \frac{d}{dt}(\Omega x) - \Omega\dot{X}\cos\Omega t - \Omega\dot{Y}\sin\Omega t \end{aligned}$$

and we rewrite it as:

$$\begin{aligned} \ddot{x} &= \ddot{X}\cos\Omega t + \ddot{Y}\sin\Omega t + \Omega(V - \Omega x) + \Omega V \\ \ddot{y} &= -\ddot{X}\sin\Omega t + \ddot{Y}\cos\Omega t - \Omega(U - \Omega y) + \Omega U \end{aligned}$$

finally as:

$$\begin{aligned} \ddot{x} &= A + 2\Omega V - \Omega^2 x \\ \ddot{y} &= B - 2\Omega U - \Omega^2 y \end{aligned}$$

The second term on the right-hand side of these equations is the Coriolis term of the apparent force, while the third is the centrifugal term, that later in the equations of motion will be summed to the gravitational force f_g.