Tidal theory

The shallow-water equations describe the motion in a homogeneous incompressible hydrostatic fluid. These equations also known as tidal theory were first obtained by Laplace (1778) showing the response of a stably stratified incompressible hydrostatic fluid under forcing. Forced tidal motion in the oceans is caused by the gravitational pull of the moon, in the atmosphere by thermal forcing of the sun. The wave equations provide a prototype model for global atmosphere-ocean wave dynamics (Müller and Maier-Reimer 2000).

Tidal theory provides furthermore a direct link to observable large-scale disturbance structures in the atmosphere-ocean system. Westward propagating Rossby and eastward propagating Kelvin waves are observed in conjunction with El Niño (Philander 1990) and decadal climate variability with its origin in the Pacific and Atlantic oceans (Latif and Barnett 1994, Grötzner et al. 1998).

1 Dynamics

We consider tidal equation on the $\beta-$plane (e.g. Gill, 1982). This fluid dynamical system is described as

$$\partial_t u = (f_0 + \beta y)\, v - g\, \partial_x \eta \tag{1}$$

$$\partial_t v = -(f_0 + \beta y)\, u - g\, \partial_y \eta \tag{2}$$

$$\partial_t \eta = -\partial_x (Hu) - \partial_y (Hv) \quad , \tag{3}$$

where t, x, y are time, eastward distance, and distance from the equator, respectively. The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation. The other parameters are listed in Table 1.

The dynamical system (1,2,3) is a generalization of Matsuno's (1966) equatorial wave equations. It is useful to introduce the new variables

$$q = (g/H)^{1/2}\, \eta + u \tag{4}$$

$$r = (g/H)^{1/2}\, \eta - u \quad . \tag{5}$$

Table 1: List of parameters used.

Parameter	description	formula	typical values
H	equivalent height		
g	reduced gravity		
R	Earth's radius		$6.371 \cdot 10^6\, m$
Ω	Earth's rotation rate	$2\pi\, day^{-1}$	$7.272 \cdot 10^{-5}\, s^{-1}$
M	zonal wave number		$0, \pm 1, \pm 2, \ldots$
N	mode number		$0, 1, 2, \ldots$
φ	latitude		
φ_0	reference latitude		
f_0	reference Coriolis parameter	$2\Omega \sin \varphi_0$	$8.0 \cdot 10^{-5}\, s^{-1}$
β	β-term	$2\Omega/(R \cos \varphi_0)$	$2.0 \cdot 10^{-11}\, m^{-1} s^{-1}$
c	barotropic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $2000\, ms^{-1}$ ocean: $200\, ms^{-1}$
c	baroclinic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $20 - 80\, ms^{-1}$ ocean: $2\, ms^{-1}$
a	meridional wave guide (Rossby radius)	$\sqrt{\frac{c}{2\beta}}$	atmosphere: $6.6 \cdot 10^5\, m$ ocean: $6.6 \cdot 10^4\, m$
t^*	time	$t\sqrt{2\beta c}$	
x^*	eastward distance	x/a	
y^*	meridional distance	$(y - f_0/\beta)/a$	
ω^*	frequency	$\omega/\sqrt{2\beta c}$	
k^*	zonal wave vector	Ma/R	

Non-dimensionalization of t, x, y yields

$$\begin{aligned}
\partial_t q &= -\partial_x q - \left[\partial_y - \frac{y}{2}\right] v && (6)\\
\partial_t v &= -\frac{1}{2}\left[\partial_y + \frac{y}{2}\right] q - \frac{1}{2}\left[\partial_y - \frac{y}{2}\right] r && (7)\\
\partial_t r &= +\partial_x r - \left[\partial_y + \frac{y}{2}\right] v \quad , && (8)
\end{aligned}$$

where the stars of the non-dimensionalized variables (Table 1) have been dropped. The dynamics (6,7,8) describe wave propagation in an inhomogenous and anisotropic medium. Zonal wave dynamics differ significantly from those in meridional direction. The primary source of inhomogenity is due to the Coriolis force. The x and t dependences can be separated in form of zonally propagating waves $\exp(ikx - i\omega t)$.

The eigenfunctions in $y-$direction are related to parabolic cylinder functions $\Big($or Hermite polynomials with weight $\exp(-y^2)\Big)$ satisfying following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] D_N = N\ D_{N-1} \quad ; \qquad \left[\partial_y - \frac{y}{2}\right] D_N = -D_{N+1} \quad . \tag{9}$$

The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics. A basic feature of $D_N \sim \exp(-y^2)$ is that significant wave amplitudes are trapped in a wave guide centered at the latitude φ_0, similar to the equator-centered Yoshida guide (Gill 1982).

The Fourier modes $\hat{\xi}_N(t) := (\hat{q}_{N-1}, \hat{v}_N, \hat{r}_{N+1})$ correspond to order $N > 0$ and wave vector k. The prognostic equations for the Fourier modes are first order in time

$$\frac{d}{dt}\,\hat{\xi}_N \quad = \quad A_N(k)\ \ \hat{\xi}_N \qquad . \tag{10}$$

and are described by 3×3 matrices $A_N(k)$

$$A_N(k) \;=\; \begin{pmatrix} -ik & 1 & 0 \\ -N/2 & 0 & 1/2 \\ 0 & -(N+1) & ik \end{pmatrix} \qquad . \tag{11}$$

Matrix $A_N(k)$ describes the dynamics of one Rossby and two gravity waves with eigenfrequencies ω (eigenvalue of $A = i\omega$) satisfying

$$\omega^3 - \omega \left(\frac{2N+1}{2} + k^2 \right) - \frac{k}{2} \;=\; 0 \quad . \tag{12}$$

The sum of the eigenfrequencies in (12) is zero due to $\mathit{trace}\,(A_N) = 0$ and

$$\sum_{l=1}^{3} \omega_l \;=\; \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \;\; \mathit{trace}\,(A_N) \;=\; 0 \quad . \tag{13}$$

For $N = 0$, the system matrix A_0 is specified to be

$$A_0(k) = \begin{pmatrix} \boxed{ik} & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & -1 & ik \end{pmatrix} \quad . \tag{14}$$

The different signs of the $\cdot_{11}-$elements in (11) and (14) originate from the requirement that the corresponding eigenmode $q_{N=0}$ in (14) is integrable (Gill 1982). This mode with $v = r = 0$ is called equatorial Kelvin wave which propagates eastward without dispersion:

$$\omega = k \quad . \tag{15}$$

The dynamics of the Kelvin wave is decoupled from the Yanai wave dynamics described by the second and third eigenvectors of matrix (14). The Yanai wave, also known as mixed planetary-gravity wave in the literature (Gill 1982), has a quadratic relation

$$\omega^2 - k\omega - 1/2 = 0 \quad . \tag{16}$$

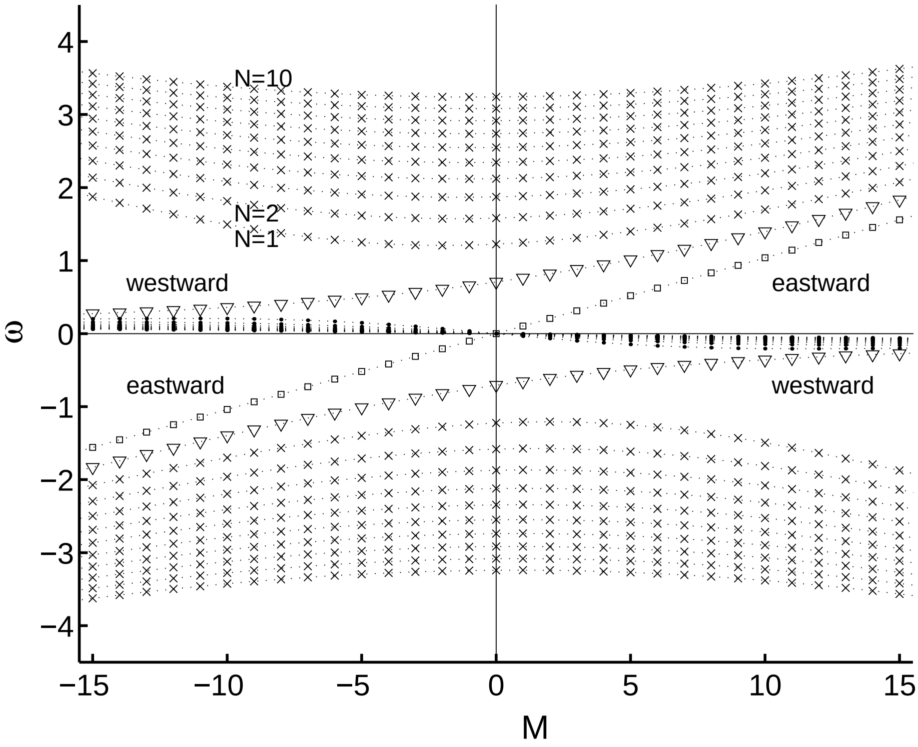

Figure 1: Dispersion relation for equatorial waves. Curves show dependence of frequency on zonal wave number M for mode numbers $N \leq 10$. Kelvin waves () propagate eastward, Rossby waves (•) westward, while gravity waves ($\times$) exist for both directions. Yanai waves ($\bigtriangledown$) behave Rossby-like for $M < 0$ and gravity-like for $M \geq 0$.

Dispersion curves for the Rossby/gravity (12), Kelvin (15), and Yanai (16) waves are shown in Fig. 1 as a function on zonal wave vector $k = Ma/R$ and mode number N. The figure depicts eastward propagating Kelvin and westward propagating Rossby modes. Gravity waves can propagate east- and westward. The Yanai wave behaves as a gravity wave for $k \geq 0$ and as a Rossby wave for $k < 0$. Note that (12) is invariant under $\omega \rightarrow -\omega, k \rightarrow -k$, which is a consequence of (13). Dispersion diagrams like Fig. 1 can be found in standard text books of geophysical fluid dynamics showing the upper (Gill 1982) or right (Holton 1992) part of Fig. 1.

Discussion and conclusions

In the limit $\beta \to 0$, the dynamics consists of gravity waves with $\omega^2 = f_0^2 + (ck)^2$. When filtering out gravity waves by eliminating the time derivative in (3), (u, v) in (1, 2) can be taken as plane waves proportional to $\exp(ikx + ily)$, where l denotes the meridional wave number. Non-divergent Rossby waves with $\omega = -\beta k/(k^2 + l^2)$ are retained only. The trapped character of the waves vanishes and the dynamics of the modes are separated for infinite Rossby radius $a = \sqrt{c/(2\beta)}$, a measure of the wave guide geography in our inhomogenous medium.

The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force (and thus $f_0 + \beta y = 0$), but would certainly observe Rossby wave propagation. In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow with non-zero vorticity flow (zonal velocity $U = R\Omega \cos\varphi$) in the inertial reference frame (Müller and Maier-Reimer 2000). Therefore, the effect of Earth's rotation is formally equivalent to a shear flow system.