5. Exercises for Dynamics I, 30. Nov. 2006

1. We consider tidal equation on the $\beta-$plane. This fluid dynamical system is described as

$$\partial_t u = (f_0 + \beta y)\, v - g\,\partial_x \eta \qquad (1)$$
$$\partial_t v = -(f_0 + \beta y)\, u - g\,\partial_y \eta \qquad (2)$$
$$\partial_t \eta = -\partial_x(Hu) - \partial_y(Hv) \quad , \qquad (3)$$

where t, x, y are time, eastward distance, and distance from the equator, respectively. The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation. Derive the dispersion relationships $\omega(k)$ for the cases:
a) In the limit $\beta \to 0$.
b) $c \to \infty$.
c) For infinite Rossby radius $a = \sqrt{c/(2\beta)}$.
d) When filtering out gravity waves by eliminating the time derivative in (3), (u, v) in (1, 2) can be taken as plane waves proportional to $\exp(ikx+ily)$, where l denotes the meridional wave number. Derive the dispersion relationships $\omega(k)$ for the so-called non-divergent Rossby waves.

2. Provide typical values of $\omega(k)$ for M,N=1,2,3 and the atmosphere and ocean.

3. Show that the eigenfunctions in $y-$direction, which are are related to parabolic cylinder functions $\Big($or Hermite polynomials with weight $\exp(-y^2)\Big)$, satisfy following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] D_N = N\, D_{N-1} \quad ; \qquad \left[\partial_y - \frac{y}{2}\right] D_N = -\, D_{N+1} \quad . \qquad (4)$$

The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics.

$$D_N(y) = \frac{1}{\pi}\int_0^{\pi} \sin(N\Theta - y\sin\Theta))$$

b) Show that the functions are orthogonal, i.e.

$$\int_{-\infty}^{\infty} dy D_N(y) D_M(y) = \delta_{NM}\, N!\sqrt{2\pi}$$

4. The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force (and thus $f_0 + \beta y = 0$), but would certainly observe Rossby wave propagation. How can this be reconciled?
(Hint: In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow U with non-zero vorticity flow.)

Table 1: List of parameters used.

Parameter	description	formula	typical values
H	equivalent height		
g	reduced gravity		
R	Earth's radius		$6.371 \cdot 10^6\, m$
Ω	Earth's rotation rate	$2\pi\, day^{-1}$	$7.272 \cdot 10^{-5}\, s^{-1}$
M	zonal wave number		$0, \pm 1, \pm 2, \ldots$
N	mode number		$0, 1, 2, \ldots$
φ	latitude		
φ_0	reference latitude		
f_0	reference Coriolis parameter	$2\Omega \sin \varphi_0$	$8.0 \cdot 10^{-5}\, s^{-1}$
β	β-term	$2\Omega/(R \cos \varphi_0)$	$2.0 \cdot 10^{-11}\, m^{-1} s^{-1}$
c	barotropic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $2000\, ms^{-1}$ ocean: $200\, ms^{-1}$
c	baroclinic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $20 - 80\, ms^{-1}$ ocean: $2\, ms^{-1}$
a	meridional wave guide (Rossby radius)	$\sqrt{\frac{c}{2\beta}}$	atmosphere: $6.6 \cdot 10^5\, m$ ocean: $6.6 \cdot 10^4\, m$
t^*	time	$t\sqrt{2\beta c}$	
x^*	eastward distance	x/a	
y^*	meridional distance	$(y - f_0/\beta)/a$	
ω^*	frequency	$\omega/\sqrt{2\beta c}$	
k^*	zonal wave vector	Ma/R	