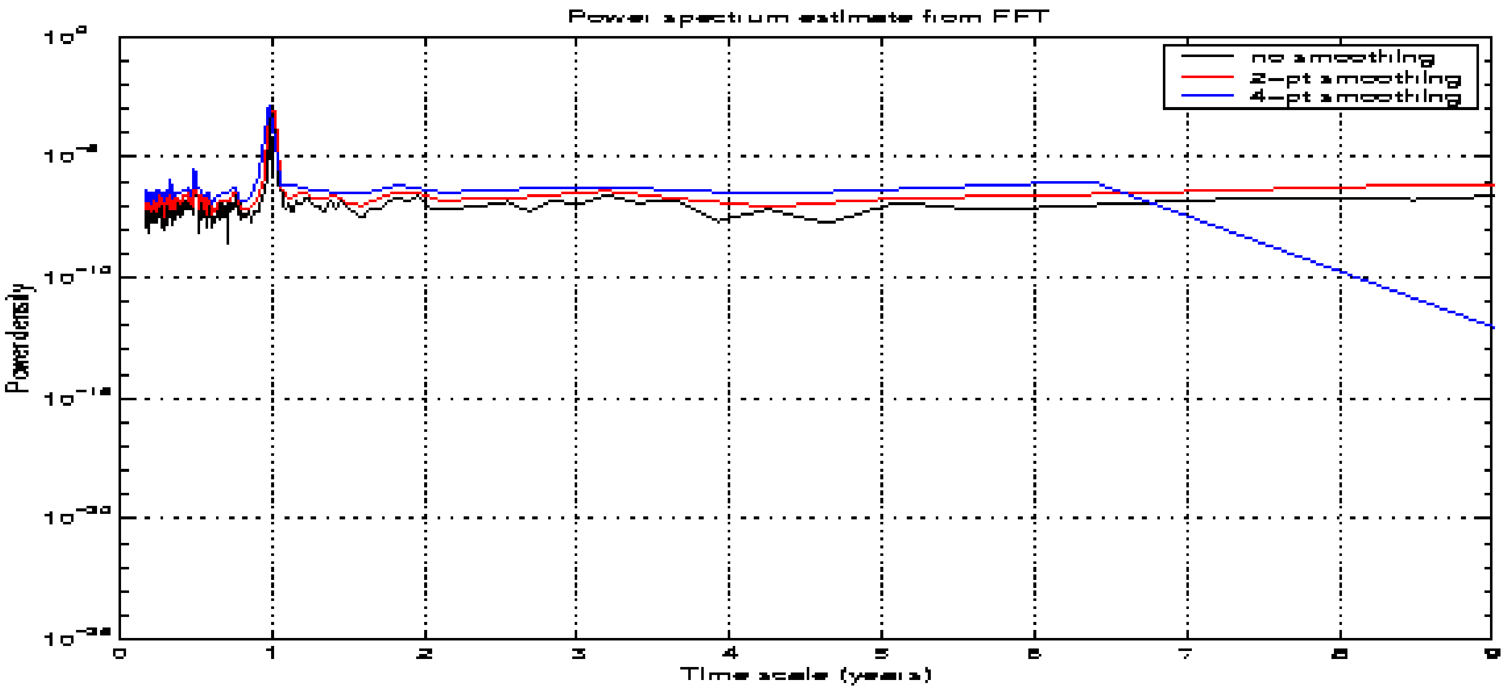
Estimating power spectra

Figure 10.2: *The FFT magnitudes of the leading PC from NCEP SLP PC.*



One way to estimate the power of just a few harmonics known *a priori* (such as the diurnal or annual cycles) is to find the best-fit coefficients between the time series and the sinusoids with a given frequency. Such a regressional analysis yields the same results as a FT for the same harmonics.

There is a good discussion on practical aspects of power spectral estimation in the *Numerical recipes* ([], 12.7, p.463). Power spectral density (PSD) normalisation: there should be some relation of proportionality between the squared amplitude of the data and the amplitude of the PSD. There are different conventions: ``sum squared amplitude'', ``mean squared amplitude'', and ``time-integral squared amplitude'', and the PSD estimators have even greater variability. It is important to never integrate the PSD outside the range of the Nyquist frequency ($[-f_c\ f_c]$).

***Periodogram*: graphical representation of the spectral coefficients:**

$$C_k = \sum_{j=0}^{N-1} c_j \exp\left[\frac{2\pi i j k}{N}\right], k = 1, ..., N-1,$$

The power spectrum is defined at $N/2 + 1$ frequencies, and the PDF can be estimated according to:

$$P(0) = p(f_0) = \frac{1}{N^2}|C_0|^2,$$

$$P(f_k) = \frac{1}{N^2}\left[|C_k|^2 + |C_{N-k}|^2\right], k = 1, 2, ...(N/2-1),$$

$$P(f_c) = p(f_{N/2}) = \frac{1}{N^2}|C_{N/2}|^2,$$

$$f_k \equiv \frac{k}{N\Delta} = 2f_c k/N, k = 1, 2, ...(N/2-1).$$

The periodogram is merely an estimate of the ``true'' power spectrum. In what sense is the periodogram a ``true'' estimator of the power spectrum? The expectation value of the periodogram estimate should on average equal the power spectrum. The std of the estimate for each bin is 100%.

One can reduce the spectral density estimation uncertainty by: *i)* taking the sum (not average) over K consecutive bins to obtain a smoother estimate of the mid frequency of those K bins (The std of the spectral estimates will decrease by a factor $1/\sqrt{K}$); *ii)* by partitioning the data sequence into K smaller subsections of equal length and using the mean periodogram from these subsequences. The std of the estimates decrease with the factor $1/\sqrt{K}$ when such measures are taken.

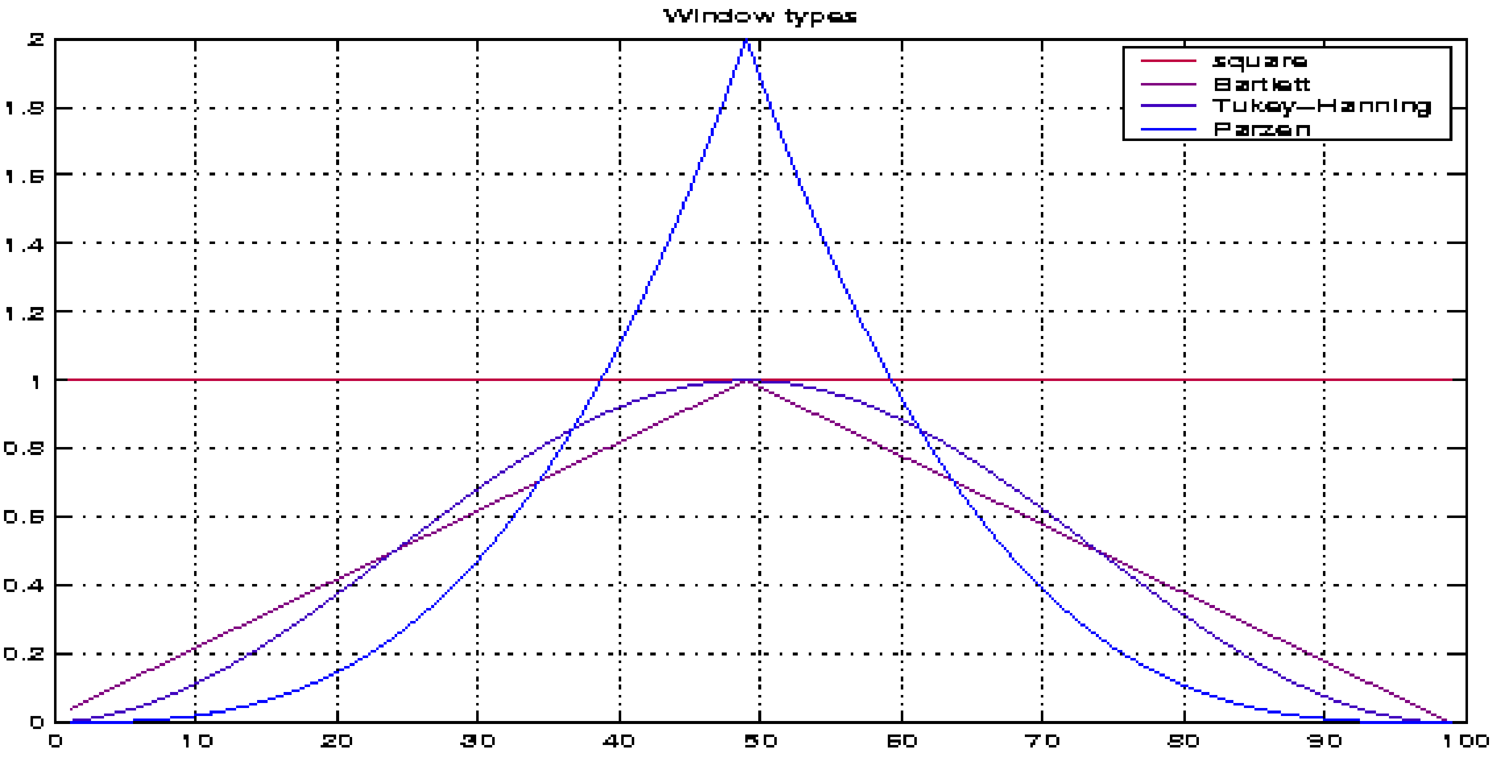
Since f_k represents whole frequency bins (halfway from previous frequency to halfway to the next), $P(f_k)$ cannot describe the continuous power spectrum $P(f)$. Expect $P(f_k)$ to be some average of $P(f)$ over a narrow window centred on (f_k) with s offset from this bin, and this window

function can be expressed as:

$$W(s) = \frac{1}{N^2}\left[\frac{\sin(\pi s)}{\sin(\pi s/N)}\right], \qquad (10.1)$$

which has oscillatory side lobes and a power which falls off with $W(s) \approx (\pi s)^{-2}$ from f_k. This function expresses the relation between the spectral estimates P_k at a discrete frequency and the actual underlying continuous spectrum $P(f)$ at nearby frequencies. Such windows extend into the adjacent frequency bins, and can cause quite substantial *spectral leakage*.

Figure 10.3: *Four commonly used window types. [stats_uib_10_1.m]*



The purpose of data windowing is to modify equation 10.1 by applying a Parzen window:

$$w_j = 1 - |\frac{j-(N-1)/2}{(N+1)/2}|,$$

or a Hanning window:

$$w_j = \frac{1}{2}|1 - \cos\left(\frac{2\pi j}{(N-1)}\right),$$

or a Welch window:

$$w_j = 1 - \left(\frac{j - (N-1)/2}{(N+1)/2}\right)^2.$$

Fig. 10.3 shows a few common window types.
